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# THE TEACHING OF MATHEMATICS

## IN SECONDARY SCHOOLS

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MADRAS
OXFORD UNIVERSITY PRESS
DELHI BOMBAY CALCUTTA
1974

*Oxford University Press, Ely House, London W.1*
GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
CAPE TOWN IBADAN NAIROBI DAR ES SALAAM LUSAKA ADDIS ABABA
DELHI BOMBAY CALCUTTA MADRAS KARACHI LAHORE DACCA
KUALA LUMPUR SINGAPORE HONG KONG TOKYO
*Oxford House, Mount Road, Madras 6*

© *Oxford University Press, 1974*

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.C.E.R.T., West Bengal
ate. 21. 3. 75
cc. No. 2447

Bureau Edn. Research
Dated 21·3·75
Accs. No 2447

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JAM

PRINTED IN INDIA
AT THE WESLEY PRESS, MYSORE, AND PUBLISHED BY
JOHN BROWN, OXFORD UNIVERSITY PRESS, MADRAS 6

# PREFACE

THIS book is an adaptation of *The Teaching of Modern School Mathematics* by E. J. James. In India the classification of schools into Grammar and Secondary Modern Schools does not exist and the range of abilities of the students in a typical Indian school is wider than in a Secondary Modern School in England. This makes some of Mr James' statements in his book inapplicable to Indian schools, yet it is an excellent book and contains material which could be valuable to teachers under training as well as teachers in Secondary Schools here. Hence this adaptation.

It may be felt that some of the topics even in this adaptation are not strictly relevant to the Indian scene. The tyranny of a strict syllabus and the financial weakness of the schools leave little scope here for experiments in teaching, and the chapters on special applications of surveying and astronomy, for instance, are very remote from what is taught in most schools. However such material will supply the teacher with the necessary background for a proper approach to the subject, and has been retained.

The chapters on Statistics and Mathematical Diversion are commended to the teachers for their careful and intelligent study.

It is hoped that the book, in this present form, will prove as popular with teachers in India as its counterpart has been in England.

R. N.

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# PART I

# *The Problems of Mathematics in Schools*

## I

## THE PLACE OF MATHEMATICS IN THE SCHOOL CURRICULUM

BEFORE a syllabus in any school subject is framed, and certainly before we start to teach it, we should decide the value and place of the subject in the education of the child. Why should we teach mathematics? What kinds of mathematics should we teach? What part can it play in the child's development? A deep and critical study of these questions may greatly influence our approach to the subject and our teaching methods. The reasons generally advanced for the inclusion of mathematics in the school curriculum may be broadly classified under two headings: (i) its utilitarian value, and (ii) its cultural value.

### UTILITARIAN MATHEMATICS

In the most narrow sense of 'utilitarian mathematics' we refer to the need for simple mathematical ideas and processes in the immediate everyday life of the individual. However, if we ask that the individual shall take an intelligent interest in the life of the community the range of mathematics required is increased. For example, many adults find difficulty in comprehending the index numbers which are used to show the rise or fall in living costs, wages, production, etc. Mathematically they are quite simple, and an understanding of them can be given to the able pupils of a secondary school. Mathematics is in many cases a means to a fuller knowledge of society, and its value in portraying social and economic conditions is rarely fully realized in schools.

In a changing world the teacher should be conscious of the new demands in the mathematics required of those who wish to play a part in the modern community. Today the urgent need is for technicians, not clerks. It is clearly impossible to cover all the various needs of these future technicians; it is possible, however, to engender in them an attitude towards mathematics which will make the subject a living instrument. Such an attitude may prove more fruitful than almost any other gift we can give them.

Mathematics is the tool of many other subjects, and in this field we find another extension of the utilitarian aspect of the subject. Geography, science, engineering, housecraft, indeed there are few subjects which do not employ mathematics to a greater or lesser degree. For example, the fundamental notions of latitude and longitude are mathematical, the geographer uses them to answer a particular need. This last aspect extends enormously the range of mathematics which can be classed as utilitarian.

The essence of all utilitarian mathematics lies in the using of mathematics. The ability to calculate accurately is not sufficient by itself to enable a child to use his mathematics. He must also have the power to decide what type of calculation is needed, and it is in this respect that much of our teaching fails. The child must be faced with situations which demand decisions from him, which need a critical judgement and an intelligent appraisal. Mathematics is then fulfilling a major educational aim, reaching far beyond the subject itself. Few other school subjects can present such a wealth and diversity of practical problems, or give such excellent training in initiative and attack to all pupils from the weakest to the most able.

## THE CULTURAL VALUE OF MATHEMATICS

While there can be no doubt that the earliest mathematics was strictly utilitarian, its cultural aspect has a very long history, extending back over 4,000 years. Indeed, all the earliest available records show this aspect of the subject well developed.

Although much of this work lies outside the scope of the child, little attempt has been made so far to show mathematical ideas in a simpler setting, so that the child can gain some appreciation of them. One of the oldest examples of 'useless' mathematics, magic squares (Chapter XVII), has considerable appeal for all but the weakest of school children. Curve stitching (Chapter XIV), a completely useless pastime, fascinates all girls. In essence it is an application of *envelopes*, an advanced mathematical topic, taken at a very low level. The game of noughts and crosses is never more than a diversion, and it has no more usefulness when played in three or four dimensions (Chapter XVII). Able pupils can, however, be led through it to one of the mathematician's strangest concepts, the fourth dimension. These are instances where mathematical ideas can give delight to pupils, and, after their first introduction, they pursue them in their own time for the pleasure they give.

## EXPRESSION

Another feature of mathematics is the training it gives in expression, a training which depends entirely on the nature of the teaching. Mathematics possesses its own language, a concise and effective medium of thought, far outstripping verbal language in its power. Training in this language should be given very great care indeed. Many difficulties arise, even as early as the Infant school stage, because the teacher fails to recognize that statements such as

$$3+2=5 \quad \text{or} \quad \begin{array}{r} 3 \\ +2 \\ \hline 5 \\ \hline \end{array}$$

are a new mode of expression for the child. Nearly always the most difficult part of a mathematical problem is the translation of a situation expressed in verbal language into its equivalent 'language' in mathematics. Once this has been done the computation which follows rarely offers much difficulty. Finally comes the reverse process; the result of the computation has to be translated back into verbal language. Children who have had deliberate training in expression (and this can be done from the age of six onwards) can attack problems at the age of 10+ with far more confidence than those who have had little such training. Such practice should be continued with older children, and this is especially important when mathematics is used in dealing with practical and social problems. It is important to stress that a correct computation does not complete the task. The result has to be re-expressed, and understood in ordinary language. These skills are not easily or quickly acquired. They need careful training over a long period of time. The long-term results, however, are most rewarding.

## THE MATHEMATICS STAFF

The present difficulty of finding mathematics specialists is common to all secondary schools. However, the lack of highly trained mathematicians need not necessarily be disastrous to the mathematics in schools. All that is needed from the teacher is a real interest in the teaching of the subject, a knowledge of the child's mind and of his ways of learning, and a true understanding of the fundamental mathematical ideas which are being taught.

It is to be hoped that in each school there is one person whose experience, qualifications, and enthusiasm render him a suitable leader of the mathematics staff. One of his responsibilities would be to assist his colleagues, who are not well qualified mathematically, in the task of teaching the subject. It is almost certain that such guidance, given tactfully, will be necessary. A suitable person, in such a position can be the source of much-needed inspiration, and may well revolutionize the work.

# II

# THE CHILD AT SCHOOL

## THE GROWING MIND

DURING the pupil's stay at school he is changing from child to adult. The physical change is easily seen; the accompanying mental changes are not so clearly defined. In a child's early life, up to the age of 7 or 8, the thought processes are very closely linked with practical experiences and all the child's achievements up to this point lie on the practical plane. In the elementary school a transition occurs, a shifting of the emphasis from the practical plane to the mental plane. From the real experiences of the child grow the mental structures which enable him to work without recourse to practical experiment. Many difficulties which the child has already overcome in the activities of the Infant school have to be faced and conquered again, perhaps two years later, when they are presented without practical aids.

Unfortunately, in many elementary schools little attempt is made to make the transfer with care. Instead of developing mental structures from real experiences, reliance is placed on the memory, and a structure of arithmetical processes is built up on memory routines. Many children are taken far beyond their ability, especially in computation. It is an educational tragedy that it is possible to turn children into mechanical calculating machines, with no idea of what they are doing, why they are doing it, or how to use their skills. In no other subject can the techniques be taught with such competence and so little understanding.

For most pupils the mathematics course must be based on practical situations with the mathematical ideas growing out of these. Indeed, for very many of our pupils we shall never get away from this; the less able the pupil, the more tied to practical considerations the mathematics course must be.

## PROBLEMS OF THE FIRST YEAR

It will be immediately apparent therefore that the task of the first year in the secondary school is of a difficult, delicate, and complex nature. The teacher will want to acquaint himself as quickly as possible with the general and individual mathematical levels of his new pupils, with the methods they understand, and with the remedial work that may be necessary. Above all he has to capture their imagination and interest in this subject at the beginning of school life. Such themes are worth a little examination.

### *Testing the Entrant*

A large number of schools give their entrants a diagnostic or attainment test in order to assess their standards. There are many arguments against this practice.

Two factors, in particular, weigh against the early formal testing of entrants. The value of any test taken immediately after a long holiday and under such conditions is doubtful. Secondly, diagnostic and attainment tests, as they are designed at present, have a restricted value. They test the child's ability to reckon; they do not test his understanding of the basic ideas, nor whether he can use his reckoning in appropriate situations. They ignore the fact that it is possible during the first weeks in the child's school life to gain his interest and to change completely his attitude to the subject.

There are many possible ways of achieving this which are open to the teacher. We must offer the child new fields to explore. Children for whom arithmetic has been 'sums' will respond with delight to number puzzles, magic squares, and other diversions (Chapter XVII). Many teachers like to give a sense of new attainment by introducing fresh mathematical studies. The beginning of a practical geometry course can be made most attractive; the first steps in algebra can be made by an emphasis on expression; graph work has a ready visual appeal; simple practical work with decimal fractions or averages might be considered. In all the work done expression should become an important factor, and the teacher must realize the various aspects of expression. A drawing in geometry, a graph showing growth, a scale model of a house or room, a diagram illustrating decimal notation, are all forms of expression for the child just as much as is the spoken or written sentence. The latter is undoubtedly the most flexible form of communication of ideas, but the other modes help the child to formulate his ideas before putting them into words. Much time may profitably be spent in the early stages discussing and writing about the mathematics being done so that the child sees more clearly its uses and purpose.

### *A Practical Test for Entrants*

The following device may be used in place of the formal tests, to test not only the child's ability to reckon, but also his ability to use computations, appropriate units of measure, etc. A set of work-cards is made, each with a question or questions dealing with a particular situation. The relevant material, measures, apparatus, etc., should be available. The children work individually or sometimes in pairs, and a record is kept of the work-cards they have completed. A sufficient number of cards should be made to ensure that, when a child has completed a task, he is not kept waiting for cards that he has not yet used. For a class of forty children about sixty cards will be found suitable. To help the teacher with the recording of work done, the cards can be classified according to the topic they treat, e.g. money, number, time, length, etc. They may also be classified according to difficulty. The children should generally be given a free choice from the work-cards not in use, but the teacher should make certain that some work is done in each of the general topics.

A simple and effective method of classification is to label those work-cards dealing with number as A1, A2, A3, etc., those dealing with money as B1, B2, B3, etc., and so on. A large record sheet shows the pupils' names and also, along the top, the numbers of the work-cards. As each card is completed by a child, it is marked in the appropriate space

| | *A1* | *A2* | *A3* | *A4* |
|---|---|---|---|---|
| Amir | √ | | | |
| Arul | | | | √ |
| Babu | | √ | | |
| Benjamin | | | | |
| Bhupal | √ | √ | | |

In compiling the work-cards two aspects should be considered: the variety of practical ways in which the child can be asked to use his mathematical skills, and the calculations which can arise from the work. For example, work-cards on time can be based on the class time-table, a railway time-table, children's ages as shown on the register, the child's own everyday life and the times taken for his various occupations, etc. A work-card on the Suburban railway time-table service involves the reading of such a table and its method of stating times. Computations based on the table will include subtraction to find running times between stated stations, multiplication to find, say, the working hours of a driver who has completed five double journeys. In work on the class time-table, finding the total time spent on any one subject in a week will involve subtraction of times to find the length of each lesson, followed by addition if they are of different lengths, or multiplication if all lessons are equal in length. Some graphical representation of the time-table may be given or asked for.

Work on money should include some counting of coins to find the total sum in a purse or box. The coins should preferably be real, but cardboard coins can be used. A 'poster' shop (i.e. a poster showing a list of goods and prices) can provide situations needing addition, multiplication, and subtraction of money. Division of money can be tested by giving the total cost of a set of classbooks or other class-room equipment. Finding the cost of supplying books and equipment to the class would entail much work and might have some effect on their subsequent treatment.

The class-room itself can be used for length measurement. Its length, width, perimeter, and possibly area may all be brought into the test; plans can be drawn, so showing the child's ability to use a scale and to work in fractions of a centimetre. The use of appropriate units in measuring will be tested. Windows, notice-boards, the child's own individual measurements, can be the sources of further work. Money can also be introduced here through the price of curtain material, or the cost of wood for the notice-board.

The class almirah or cupboard can provide ample material for counting and work in number. Counting the number of textbooks in each set, finding the total number of books used by the class; finding the number of writing-books or sheets of paper in a packet, and counting the number of packets to find a grand total, are situations which make a very real use of number. Various school statistics such as attendances, numbers of holidays, etc., can be used.

A surprising range of arithmetical topics and processes can be covered in this way. Much previously learnt work is tested and revised in a refreshing manner, yet the child in no way feels that he is being tested. As the work proceeds, the teacher will note the child's ability to express himself, and also the extent of his mathematical vocabulary. Many of the situations can lead easily and naturally to the introduction of new ideas, and the child's reception of these can be seen. For example, the idea of an average follows quite readily from the inspection and comparison of the heights or weights or ages of a group of children, or of the whole class.

The preparation and organization of such practical work obviously needs much time and careful thought, but it does not, of course, need fresh preparation each year. Such an activity can be used at other times and may well be put into operation several times a year.

## THE MIDDLE YEARS

If the task of the first year is done well, the subsequent years should offer little difficulty. There is a belief that a growing dislike of mathematics through the years of the secondary school is inevitable. Many teachers, including head teachers, feel that this is a natural distaste for the subject, having felt it themselves. Yet, mathematics can gain an ever-increasing hold on the children. The distaste is for the traditional manner of presenting the subject, as teachers bold enough to experiment with other approaches soon find.

# III

# METHODS AND APPROACHES

## GENERAL CONSIDERATIONS. 1. FOUNDATIONS OF PRACTICAL EXPERIENCE

WHEN considering teaching methods two fundamental points need to be emphasized. Firstly, the distinction between a mathematical idea and the calculations based on that idea must be clear in the teacher's mind. Secondly, for many pupils a knowledge of a mathematical idea can only arise from some practical situation.

With these factors in mind we can move towards a general method to be employed in our teaching. The concrete and practical situation must be the means by which a new mathematical idea is introduced. Such situations should be carefully chosen both for their mathematical content and for practical use. Before the new idea can be made the basis of routine or abstract calculation, material and time should be provided to ensure that the idea is absorbed.

Wherever possible, formal definitions and the use of technical terms should be delayed until the basic idea is seen. In this way the definition clarifies and expresses an idea which is already known. Similarly, when a simple rule states a process of calculation, the rule must not be given until the children see the process clearly, and the rule expresses the method they are already beginning to use. If a mechanical rule is given too soon, it dispenses with any need for understanding. It is a quick way of finding the answer, even if the child does not understand what he is doing.

Many teachers hesitate to spend much time on simple basic work; they feel they must hurry on to computation in order to have 'something to show'. Mathematics teaching has always suffered from such impatience. Few have had the courage to delay a topic until a child is ready for it. Those who have done so have been fully rewarded by the phenomenally rapid progress made subsequently by their pupils.

## 2. REVISION OF WORK

The need for constant revision is nearly always an outcome of approaching a topic too soon or of hurrying over the fundamentals. Ideas should not need revising; their introduction should be such that they are known completely. It is reckoning that will need revision from time to time. There seems little point, however, in setting revision examples in a procedure only because pupils have forgotten this particular type of calculation, and still less point if they are then allowed to forget it all over again. If a careful watch is kept on the work done by the class, uses can be found for most of the processes known, so obviating much of the need for special revision exercises. In this way pupils retain their computing abilities through usage, rather than regain them by revision. When it is difficult to find a use for any particular computation in the schemes of work, the value of that computation may well be questioned.

There is a case, however, for allowing the children to forget some of the mathematics they have done, or rather, for not insisting that they remember it. We do not ask that the 16-year old pupil leaving school shall remember every poem he has learned in his school life, but we hope that the learning of poems has helped in his cultural development. In the same way some mathematical studies can be of value in the development of the child without being necessary to him when he leaves school.

Revision can be carried out in an interesting and lively manner through an activity such as is described in Chapter II, or by a topic (see Approaches 2). The range of situations used can be chosen to cover the types of work to be revised. Sets of graded mechanical computation cards, or the practice exercises given in some textbooks, should be available to provide the necessary practice for individuals who need it. It will frequently be found that when the activity discloses a weakness in the child's mathematical skills he will of his own initiative select and work at the graded practice cards.

## 3. PATTERN IN MATHEMATICS

One essential feature of mathematical thought is the evidence of pattern. The pattern is not a visual one; it is a mental pattern, the mind seeing a regularity of behaviour. For example, when finding the area of a rectangle drawn on squared paper the common pattern is established when the child no longer counts all the individual squares, but counts the number of rows and the number of squares in a row.

Progress in mathematics depends very much on the ability to see such underlying pattern, and this ability grows with the maturing child. It is rarely found in the young child. It can be encouraged by careful teaching, so that the child begins to look for pattern early in his approach to a new idea.

Besides encouraging the child to look for pattern on all possible occasions, other methods are open to the teacher to illustrate pattern. An obvious and highly important one is the use of graphs. These present the patterns in a visual form, and visual patterns are generally easy to recognize. Merely seeing the graphical pattern is not sufficient by itself; the child should then state what the pattern tells, i.e., state the mental pattern from an observation of the visual pattern.

Tabulation is also valuable and much use should be made of this in the school. Sometimes information should be presented in a table; at other times a table should be compiled from information gained. The pattern is not so easily seen in a table of information as in a graph, but the table is nearer the mental pattern. It presents the information in an orderly manner so that the pattern is more readily seen. Few children find difficulty in seeing the pattern in the time-table of a regular bus service. (See also the end of this book for number patterns.)

## 4. INFILTRATION

There are many mathematical ideas with which the child has acquaintance through his everyday experience. He uses ideas of rate, ratio, and proportion frequently; if he gets 6 sweets for 10 p. he expects 12 sweets for 20 p. He almost certainly knows horizontal and vertical directions, and has some idea of the relationship between them, although he probably does not know the mathematical terms. There are many such ideas which the child uses, but he is rarely aware of the fundamental idea he is using, and certainly cannot express it. With notions such as these it is worth while to introduce their mathematical treatment slowly over a long period, using examples which point to clear cases. Then, when the child has absorbed the idea through usage, he is ready for a more formal treatment, a mathematical treatment which clarifies and expresses something which is already known within. Thus, ratio and proportion ideas can be used in simple contexts in the early stages.

This method of *infiltration* is most rewarding, but clearly needs careful organization of syllabus. We must always be looking forward to new ideas and seeking opportunities for showing them in a variety of situations, so that they are gradually absorbed over a period of time. Further comment is made on this point in the section on making a syllabus (Chapter IV).

Infiltration can be used for ideas other than those in common use. The first introduction to directed numbers may be through comparing individuals with an average. If the heights of all the children in a class are measured and the average found, each child can state its own height as, say, 5 centimetres above average, or 3 centimetres below average. This is clearly well within the ability of most school children, and the use of 5 centimetres above or 3 centimetres below gives a first sense of directed number. The abbreviation to +5 cm and —3 cm will follow later quite naturally and easily.

## 5. DEVELOPMENT THROUGH USAGE

The method of infiltration uses ideas before their mathematical treatment and definition. In many cases the development can be through usage also. Much of the treatment of graphs can be in relation to other work. A knowledge of graphs is built up, not by studying graphs only, but by using them as suitable opportunities occur.

There are many such mathematical notions and procedures which, while they are important in themselves, should be taken as auxiliary and not major teaching topics. Ideas of approximation, accuracy, estimation, significant figures, decimal places, to quote but a few, are much better shown and developed when occasion arises in connexion with other work. A lesson devoted especially to approximation is bound to smack of artificiality. Yet whenever a child measures a line with a ruler he is using approximation although he does not realize this. The idea that we can never measure the length of a line exactly, but only to a stated degree of accuracy is not an easy one, only the more able of the school pupils will see it. The idea of measuring to the nearest half of a centimetre is quite an easy one, however, and children are quick to see and use it. Estimation to, say, 0.10 cm (i.e. a millimetre) follows later as accuracy and the child's knowledge of decimal fractions increase.

## 6. ANALYSIS AND GRADING OF WORK

It is not intended to deal with procedures of calculation, and their analysis and grading, to any extent in this book. So much has already been written on these points that it serves little purpose to add more. It is valueless, however, to grade difficulties in computation if the child does not understand the fundamental idea. You may analyse the difficulties of subtraction of number with the greatest care, yet the child is no better off if he has not first of all had sufficient practical experience of subtraction situations to know when to use this type of calculation. The teacher knows only too well the question, 'What kind of sum is it?'

The first essential is to ensure that the child knows what he is doing; for many pupils this means seeing and using the idea in some real or practical situation. Then, when we base reckoning procedures upon the idea, is the time for thinking in terms of graded computation.

## 7. Class Discussion

An important part of work in mathematics should be the discussion which arises in various ways. The wise teacher uses it to great effect in formal lessons, and it is equally valuable in other approaches. Through discussion at the start of a piece of work the real issues are brought out more clearly, stated probably in the child's own language. Discussion as the work proceeds and discussion at its conclusion are both necessary to clarify what has been done. There is always the danger in practical mathematics that the practical circumstances hide the fundamental mathematics for some pupils. They work intuitively, achieving a satisfactory result without being conscious of the mathematics employed. Perhaps the most important use of discussion is to ensure that this does not occur.

The discussion at the end of a piece of work should be of especial value in getting the children to express what has happened and what conclusions can be drawn. The importance of expression has already been stated. For children who have had little practice in written expression, discussion is essential. The teacher must get them to describe orally first of all, before asking them to write.

The examples given below illustrate the use and value of discussion.

(*a*) *Long Multiplication.* If an example such as 35 × 23 is set to a class, it is possible that both of the following methods of working will be found.

$$\begin{array}{r} 35 \\ 23 \\ \hline 105 \\ 70\phantom{0} \\ \hline 805 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 35 \\ 23 \\ \hline 70\phantom{0} \\ 105 \\ \hline 805 \\ \hline \end{array}$$

There will be many occasions when the teacher has to work such an example on the blackboard, and discussion of the two methods early in the course will avoid the confusion which may very well arise if the problem is ignored. Even for an able pupil the process of long multiplication may have become so automatic that he is no longer aware of the operations being carried out. When multiplying by 23, we multiply by 20 and by 3, and add the results. It is immaterial whether we multiply by 20 first or by 3 first. Thus it is seen that there is no fundamental difference between the two methods; the difference lies in the order in which the operations are carried out.

(*b*) *Directed numbers*. See Chapter V.

(*c*) *Sampling*. The idea of statistical sampling can be shown in many ways. One of them is described in the work on a car census in Chapter XV.

## 8. THE MATHEMATICS LIBRARY

One aspect of the intellectual growth of the schools in India is the increasing importance given to the school library and its use. A good mathematics section is vital to the well-being of the subject in the school. The choice of books must not be confined to school textbooks; there is a wide range of other books suitable for a mathematics library and which add much to its value. Some textbooks not normally used in the school can be included; a prominent development in some modern mathematics textbooks is their much increased ' readability '. Among other types of books should be those dealing with particular uses: astronomy, surveying, drawing, practical geography, etc.; suitable source books of statistics and other information: *India—Annual*, published by the Publications Division, Government of

India; *Hindustan Year Book* by Sircar, etc. There must also be a few books on the history of mathematics. Some suitable books may be in other sections of the library (geography, science, etc.). A list showing such books and where they are to be found can be displayed near the mathematics section.

On the whole the books in this section of the library will be used for reference rather than for general reading. This means that we can include in our selection some books which appear far beyond the scope of the school child. No one expects a pupil to read Hogben's *Mathematics for the Million* in its entirety. As a reference book it is a valuable source of much interesting information and, as such, can be used by able pupils.

A list of books is given at the end of this book.

## 9. Using a Textbook

The usual purpose of a textbook is to provide a suitable course of work, together with an adequate supply of examples. At this early stage in the development of many schools, textbooks can also be source books of ideas for the teacher. Some of the more recent textbooks stimulate the interest of the pupils by the use of practical situations, puzzles and diversions, the history of the subject, etc. In many cases topics may be further developed by the teacher, and quite frequently they can be related to the children's environment or experience. The work then acquires a vitality which no textbooks by itself can give.

Sometimes the pupils should be asked to make up questions based on the new material. Even where fresh material is not introduced, benefit is obtained from making up additional questions. In this way the pupils take a more active part

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in the mathematics period and a livelier interest is aroused. The practice gained in expression in work of this kind is valuable; even more valuable is the insight given to mathematical ideas. Children can often answer questions correctly without being fully conscious of the fundamental ideas they are using. By reversing the process and asking the children to compose the questions we are forcing them to a greater awareness. It is quite true, of course, that children should be able to carry out simple processes automatically. This does not mean, however, that they should be unable to think of the processes involved when necessary. At times it is essential that they should do so.

When the work is carried out in this manner each class can compile its own book of examples. The examples chosen to enter in the book should be those of especial merit or interest, particularly those based on the class itself, the school and the locality, and items of topical interest. Over the period of a few years these books can prove of very great value to the teacher.

Even when children are working directly from a textbook, the work can be enlivened by encouraging the collection of relevant material for illustration. Work on the English money and units of measurement can be illustrated by pictures of England, English stamps and money, books with price marked in English money, maps of routes to England, ship and air tickets, etc.

## APPROACHES

For very many years the formal class lesson—a period of instruction and possibly discussion, followed by the pupils working examples—was the only kind of lesson given in mathematics. In some schools it is still the only kind of lesson, but many have experimented with other forms of

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approach and with varying degrees of success. One fact must be stated immediately. The formal lesson is not to be decried. It has its own value and its own place; but it is only one of many possible ways of teaching mathematics.

It is intended to give in the remaining part of this chapter outlines of other approaches. All are dealt with in greater detail in the particular examples given in Part III, but what is needed here is a general picture of the methods. No one approach is better or more important than the others; the wise teacher achieves his ends by a variety of methods.

There is a feeling among many teachers that the practical approaches, while having possibly the merit of greater interest, do not achieve the same high level of attainment as the formal lesson. This feeling, however, is wholly unjustified. Practical approaches have in many cases given rise to work of the highest quality.

We should remember that the formal lesson has a long history of development. To condemn the practical approach after one brief and unsuccessful experiment is surely premature. We must seek success and not expect to find it without effort. It is obvious, too, that new methods demand new class-room techniques. In many cases failure with the new methods has been due, not to some defect in the methods themselves, but to a failure on the part of the teacher to provide the requisite modifications in class management and organization. It is for this reason that, in some of the examples given in Part III, methods of class organization are detailed.

### 1. Activities

The term *activity* has been the centre of bitter controversy during the last two decades. Let us start by defining *activity*. It is some physical action on the part of the child which acts as a stimulus or aid to mental development. It is the latter part of this definition which has sometimes been ignored by the over-enthusiastic. An activity is valueless if it possesses no intellectual stimulus, and the work should be guided to ensure some mental growth on the part of the child.

Paradoxically, it is sometimes the activity which is most carefully organized and guided that gives least to the child. Surveying is an excellent example of an activity, although it is also included under another heading in this chapter. One of the merits of practical surveying lies in its abundance of simple concrete problems that are well suited to the learning child. All too often the teacher gives more guidance than is necessary, so solving the problem before the child attempts it. He admittedly gets a higher percentage of correct results and in a shorter space of time, but he is losing most valuable training in ' attack '.

For example, when first measuring a playground or other area with four sides, few children realize immediately that the four sides alone do not fix the shape; a diagonal or angle is also needed. It is much better that the pupils should err in a case like this, providing they correct the mistake, than that the teacher should prevent such errors by detailing carefully all measurements to be made. If a mistake has been made and then put right by the children, they have been forced to think about and to understand the situation. On the other hand, if the teacher has prevented such errors, the correct result may have been obtained merely by following the directions given and not necessarily through understanding. The liveliness of response and independence of thought gained by using these challenges to the child's intellect far outweigh the doubtful merit of more 'all-correct' answers.

The form of class activity described earlier in the book is valuable where it is impracticable for the whole class to be doing the same kind of work at the same time. By the use of individual work-cards detailing tasks to be carried out, there can be considerable saving of material and apparatus, while organization is made easier. It is not necessary that all the class should be physically active at the same time. Many of the cards could set tasks entailing no movement on the child's part, e.g. work based on a local map with questions on scales, distances, routes, etc. This point is further developed in the next section on topics.

The actual carrying out of operations as performed in an activity is a most valuable help to the weaker children. The written sum then becomes a record of actual happenings and so acquires significance and meaning, perhaps for the first time. It is surprising how much some of the weaker children can do in realistic situations, especially those of a familiar nature, such as shopping.

Details of an excellent organized activity, the *Money Fair*, are given in Chapter XI.

## 2. TOPICS

The topic is a form of approach which has become increasingly popular in the last few years. It is sometimes termed a project, but the true project is much more ambitious and all-embracing than the mathematical topic. In a 'topic' a field of study, such as the local bus service, is taken and treated mathematically. It is a most rewarding form of approach in the school, capable of great variation in the range of studies carried out, in forms of organization, and in the levels at which the work can be done. It shows mathematics in use and appeals to the practical mind of the child. The child is not physically active as in an activity, although the two may often combine. Nevertheless, we are treating real situations, and the same kind of stimulus exists.

One important aspect of topic work needs to be stressed. It should give the child something more than practice in arithmetic. If, for example, farming is taken as a topic the work done should give the child some knowledge of farm life. Thus the full aim of a topic extends beyond the mathematics employed in it. Mathematics becomes the medium through which the child gains an insight into various aspects of life. It is a waste of valuable educational opportunities to use farming statistics, weights, measures, etc., merely to provide the child with calculations. In a topic there should be a sense of development and a final feeling that something more has been acquired than an increased control of mathematical skills. It is this aspect of the topic which has not been fully realized yet; frequently the basis of study is used only to give practice in arithmetic.

This particular merit of the topic, its ability to teach something more than mathematics, becomes highly important when we deal with what may be called *social arithmetic*. Topics such as a comparison of living conditions over the last fifty years, India today, the nation's health, the industrial development of India under the Five Year Plans etc., offer ample scope for mathematical skills, but their value can clearly be extended much beyond the mathematics involved.

The work done in a topic may demand a new idea or process. Some teachers avoid this, feeling that difficulty would arise unless the idea had been taught and practised formally. Yet no better learning situations can be devised than those which show the need for a new process or illustrate the new idea. The first and major difficulty of teaching any new idea—showing what it is, and how and why it is used—is overcome in the most natural manner.

*Topic work and class organization.* There are many ways of organizing topic work with a class. The simplest form for the teacher is where the pupils are given the necessary information, and they answer a series of questions which use and develop the information. A topic need not be carried out by answering set questions, however, and it may gain thereby. A study of daily newspapers can produce a stimulating two or three weeks' work (Chapter X). In a topic like this the work done arises from class discussion, with the pupils feeling that they are making the decisions.

When the total amount of work to be done in a topic is large, the class may be divided into groups, each carrying out its own particular tasks. It is useful training for one member of each group to act as a leader. He is the one who should contact the teacher when the group finds itself in difficulties, or is uncertain what to do next; also he reports progress each day. When the work is completed, the group leaders should tell the rest of the class of the work done, the information found, and the conclusions reached.

Topic work may also be carried out by pupils working individually or in pairs. For example, a leaflet giving details of a touring holiday by bus, together with appropriate maps and question cards, can provide work for one or two pupils at a time. A ' library ' of similar individual topics may be built up, and will be found an enjoyable alternative to the class lesson. The class activities described in Chapter II are probably best organized as a combination of activity and topic work.

The importance of the mathematics library has already been stated; much topic work can be based on information to be found in the books of this section. The training in searching for information is most valuable, and the pupils should be able to use reference books such as *India—Annual*, *Hindustan Year Book*, *Current Affairs* etc. Some topic work of a descriptive nature may be asked for. A history of Indian coinage, or a description of some of the calculating methods and devices used some centuries ago, will send the pupil to the library looking for suitable books. Some help from the teacher will probably be needed in many cases.

Details of some topics are given in Chapter X.

*Marking group and individual work.* It is obvious that the marking of topic work makes fresh demands upon the teacher's skill and time. Individual work can provide the teacher with an impossible task if normal marking methods are used. Where a library of individual topics has been compiled, answer sheets may be made when the question cards are made. Self-marking by the children can then be used to a considerable extent, although the teacher should see and initial all work done. There is often no need for anything more than a brief scanning of the work to ensure that the essential points are covered. When group work is being done the members of the group frequently act as a check upon each other's calculations. The teacher's task is to make sure that the group are working along the right lines, and that the calculations being done are necessary. He can accept that they are correct if most of the group agree, provided, of course, that the members of the group work independently.

## 3. SPECIAL APPLICATIONS

We have already mentioned the wide use of mathematics in other subjects and in particular fields of study, and the value of these special applications in the school. A few examples are: *surveying; architecture; navigation; mathematics of astronomy; mathematics of geography; drawing; mechanics; statistics.*

It is important when dealing with such special applications to realize the reason for their use in the school. They show mathematics in use in a wide variety of ways, but it is essential that the pupils see the underlying mathematical ideas which are being used. It is often possible to teach these applications so that the pupils become skilled in their procedures, but gain little or no true mathematics. In surveying, the primary aim is to use the situations arising from surveying as a means of teaching and using a number of fundamental mathematical ideas. That the children become skilled in the procedures of surveying is of secondary value. For this reason the teacher will often discard modern professional methods and conventions (even if he can afford modern professional apparatus). He will use simple methods and simple apparatus which enable the pupils to see the fundamental principles more clearly. Many of the old navigational instruments (Chapter XII), the astrolabe, Jacob's staff, and geometric square, have the merit of simplicity in design and construction. They also possess the fascination of their history; these are the instruments used by sailors when they made their first voyages round the world some hundreds of years ago.

Single isolated lessons may be given showing brief aspects of these special applications, or a longer course may be planned. For example, a very brief study of the turrets and spires of the Moslem period could be done in a few lessons. On the other hand, it is quite easy to occupy a double period a week for four years on surveying or drawing.

Further information on some special applications is given in Chapters XII-XIV.

## 4. HISTORY OF MATHEMATICS

Some attention should be paid to the history of mathematics. The subject has a fascinating history and many mathematics lessons can be considerably enlivened by reference to it. The old navigational instruments have already been mentioned, and in many ways the historical instruments and techniques are well suited to school children. Each school should possess some reference book giving details, especially of the early history of the subject. (*The History of Mathematics*, Vol. II, by D. E. Smith, published by Ginn, is outstanding.)

Besides the occasional use of history at appropriate points in work, historical topics may be followed. The history of the measurement of time provides an excellent range of work from the experimental making of water clocks and sand glasses to the trigonometry involved in the calibration of a sundial, or the study of a pendulum and its formula. A study of early number systems serves to emphasize the structure of our own counting system.

Apart from these uses, some knowledge of the history of the subject is valuable in the way it shows mathematics as an ever-developing study, and not merely a static collection of ordered facts.

## 5. PUZZLES AND DIVERSIONS

One of the earliest recorded pieces of arithmetic is the magic square, showing how man over 4,000 years ago saw in the subject a means of intellectual entertainment. This aspect of mathematics is often regarded as out of place in the present-day school, since it is not 'useful' mathematics. It can, however, be of great value, if only in the practice provided in the basic skills. Moreover, it is surprising how interesting most pupils find such diversions, and their inclusion in a course invariably meets with an enthusiastic response. Frequently they are of considerable mathematical value. For example, from the apparently useless pastime of tangrams a surprising knowledge is derived of the simple properties of the shapes used.

Details of some diversions are given in Chapter XVII.

# PART II

# *The Content of the Course*

## IV

## FORMING A SYLLABUS

It is not proposed here to give a recommended syllabus. We will confine ourselves therefore to the basic considerations governing the construction of a syllabus.

Although methods, usages, and applications are many, the fundamental mathematical ideas and principles are more restricted. These are the threads running through the work. A syllabus should be based in the first case on the mathematical notions it is intended to teach, together with the uses of mathematics that the future citizen will need. There is an obvious minimum content which forms the core of most school syllabuses, and this minimum should be decided not on the ability of the weakest pupils in the school but on the essentials that a child will later need to be able to share and interest himself in the life of the community.

When the first framework has been selected the teaching methods to be used, topics, etc., can be considered. Some indication should be given of the types of practical situation which can develop particular new ideas or calculating methods. The choice of teaching methods, etc., may call for some reconsideration of the outline syllabus: a rearrangement of order, for example, may allow one topic, chosen to illustrate a particular mathematical notion, to be used as an introduction to another mathematical notion.

Careful planning at this stage can introduce both infiltration and auxiliary usages (Chapter III) with their consequent advantages. After the main features of the syllabus have been established, reference forwards and backwards may be made to find suitable points of infiltration. Similarly an examination of the work to be done will suggest opportunities for developing auxiliary usages.

Two examples follow to illustrate the points made so far on forming a syllabus. A scheme of work for *Trigonometry* shows the development of a particular mathematical subject, while an example of the topic approach, *Farming*, illustrates the range of mathematical ideas and usages which may be covered by this method. Thus in one case we show a mathematical idea slowly developing over the whole four years; in the other we show a wide range of mathematics employed during a few weeks' work.

*TRIGONOMETRY.*† Before the subject of trigonometry is introduced, it is necessary for the pupils to have a knowledge of decimal notation, including multiplication and division by decimal fractions; ratio; and elementary properties of triangles, including similar triangles. It is advisable that the following have also been treated: use of tables; scale drawing; simple algebraic use of letters; graphs.

The following list shows a possible placing of topics:

*Stage One.* Scale drawing; first ideas of ratio; first ideas of similarity (ratio and similarity are both used in the shadow stick method of finding heights, Chapter XII); graphs of continuous functions (e.g. length of shadow of stick throughout a day); decimal notation, first decimal place.

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† See also Chapter IX.

*Stage Two.* Extensions of previous work; formal treatment of ratio; multiplication and division of decimal fractions by whole numbers; recognition of similar figures; use of tables (e.g. table of squares of numbers); introduction of algebra.

*Stage Three.* Properties of similar triangles; multiplication and division by decimal fractions; introduction of logarithms; the tangent ratio, solution of problems using the tangent ratio.

*Stage Four.* Further work on logarithms including negative characteristics; Pythagoras' theorem; solution of right-angled triangles using tangent ratio and Pythagoras' theorem; sine; cosine. (Further extensions may be made with very able pupils. See Chapter IX.)

It is clear that much of the above work will be taught even if trigonometry is not included in the syllabus. Other requirements may place some of the topics earlier in the course.

*FARMING.* A *farm topic* could include the following: Agricultural measures.

*Reading maps:* use of scale; areas in acres and hectares; areas of rectangles, triangles, parallelograms, trapezia, quadrilaterals. (Some of these areas may be expressed as formulae.) Percentages of land used for grazing, growing hay, wheat, paddy, etc.

*Growing wheat:* estimating costs of ploughing, seed, fertilizer, etc., cutting and stacking, threshing. Rate of working. Volume of stacks. Yield of wheat per hectare. Balance sheet showing cost of growing wheat, and receipts from sale of grain and straw.

*Dairy cattle:* grazing requirements. Records of milk yields of herds and of individual cows shown in tables and diagrams. Sale of milk. Growing hay; maintenance and production rations. Percentage compositions of milk, butter, cream. Yield of butter and cream from milk.

An ideal syllabus should never be rigid, but should permit adaptability especially in approaches and applications. Fresh topics, introduced from time to time, also help to preserve the interest of the teacher. It must be remembered that a stimulus provided to the teacher is bound to communicate itself to the class.

## A UNIFIED SYLLABUS

It is usual to divide the syllabus into the components, arithmetic, algebra, geometry, trigonometry, etc. For some years now there has been an endeavour to weld these into one so that the pupil regards mathematics as a single subject instead of a set of isolated subjects bearing little relationship to each other. It may be found easier in compiling the syllabus to classify the work to be done into algebra, arithmetic, etc. Would it not be better to include in the time-table only the subject *Mathematics*, rather than devote certain periods each week to any particular branch of the subject?

The teaching of the weaker children has its own particular difficulties in mathematics as it does in most other subjects. It is probably here that practical methods are of the greatest value. It has already been suggested that the mathematical development of the child falls into three stages (Chapter II). The weaker children will obviously be still in the transitional stage or, in some cases, still at the stage of practical achievement only. With the truly retarded child, it is probable that he will never acquire any mental skill, but that all his work must be related to practical situations.

It must once more be stressed that the work done should contain some practical operation to be carried out, and which either explains or performs the calculation needed. The using of real situations as a source of written examples is not by itself sufficient. The question, '*Mani spent 25 p. in one shop, and 13 p. in another shop. How much did he spend altogether?*' cannot provide the same learning experience for the backward child as the actual performance of the same operation in a shopping activity. The sum becomes a record of what the child has done. It will be found with these backward children, as with young children, that practical ability is far greater than mental ability.

The range of work possible with weaker children is by no means as limited as some people believe. A shopping activity of some form or another is an essential part of such work. If the *Money Fair* organization is used, the operations can include the reading of simple time-tables, the reading of simple maps or plans, measuring, etc. Drawing and mensuration in simple forms are also suitable, and from them can arise a very elementary geometry. Model making of various kinds with the need for accurate measurement, work in fractions of an inch, scales, etc., can provide excellent work for the weaker children. Full use should also be made of other practical school subjects: gardening, domestic science, handicraft, etc.

The rest of Part II is given to a survey of some of the more important mathematical ideas, etc., which should be considered when compiling a syllabus.

The reader should also refer to *The Teaching of Mathematics in Primary Schools*, a report issued by the Mathematical Association of Great Britain and published by Messrs. G. Bell. Although this is intended for teachers in primary schools it is of inestimable value to the secondary school teacher also, and especially to those who have to deal with the less able children. It is an authoritative statement on the mathematics which should be taught in primary schools; much of this mathematics will obviously be in use in secondary schools.

The following are some other reports issued by the same Association which are recommended:

*The Teaching of Arithmetic*
*The Teaching of Algebra*
*The Teaching of Geometry* (second Report)
*The Teaching of Trigonometry*
(All published by Bell).

# V

# ARITHMETIC

## 1. NUMBER

THERE is very little mathematics in schools in which number does not play an important part. The four rules in number form only a part of the treatment which should be given if children are to use their numbers effectively. Initially this chapter will deal with such notions as position value and number structure, magnitude and order.

Our system of counting in tens, hundreds, etc., is a decimal notation. Those quantities generally referred to as 'decimals' should be termed 'decimal fractions'—fractional parts written in the decimal notation. For the Romans, using an older number system, X was always a ten and C was always a hundred; for us, our figure 3 may denote three, or thirty, or three thousand, or one-thirtieth according to its position in a row of figures. Children must be perfectly certain that the 7 in 279 means seventy, that the 6 in 26,905 means six thousands, and that the 4 in 3,934 means four.

All ancient number systems needed some form of calculating device or abacus, which provided the position value their systems lacked. For many children in our schools, work with an abacus can give another, and practical insight into number structure and simple operations with numbers. The Indian abacus, a wooden base with vertical rods or wires, is the simplest form to make (Fig. 1).

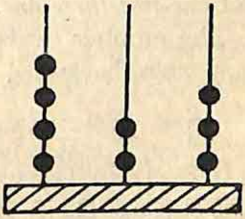


FIG. 1

Large wooden beads form suitable counters. (See also Chapter XVI).

Work on graph paper shows in a visual form the small unit squares, columns of 10 units and squares of 100 units (Fig. 2). The pupils should be asked to recognize and read numbers represented in this graphical form. Those who do the work slowly will almost certainly be found to be counting single squares instead of counting the columns as tens, etc. Pupils should also shade or colour graph-paper squares to represent given numbers.

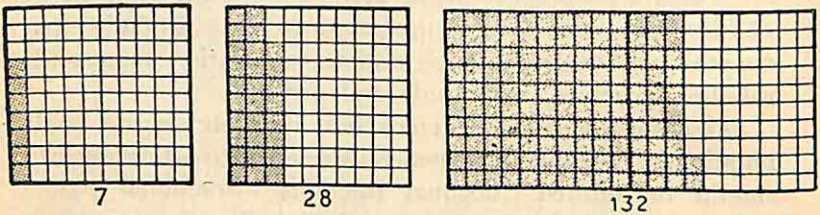


FIG. 2

The use of the abacus and the squares of the graph paper are illustrations of the decimal system. The child should be able to use the decimal system without them, but they may help to form the correct mental structures. The teacher can help by never stating a number in terms of figures only; 1,531 should be *one thousand, five hundred and thirty-one*, not *one-five-three-one*.

The number cards devised by Mme Montessori (Fig. 3) may help backward children to see the construction of

1 2 3 4 5 6 7 8 9
10 20 30 40 50 60 70 80 90
100 200 300 400 500 600 700 800 900

FIG. 3. Montessori number cards

numbers. To show the number 375, the 300, 70, and 5 cards are used, placed upon each other as shown in Fig. 4.

Counting in tens or hundreds is useful. Very often calculation is considerably eased by the ability to see 2,500 as 2½ thousands, or 25 hundreds, or 250 tens. Counting bundles of 10 objects, or boxes containing 100 objects, and writing the answer in words and in figures is good practice. Individual work-cards may be used in this and other cases so that the pupils are not all using the same material.

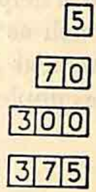


FIG. 4

Magnitude is not the only aspect of number; order is also important. Arranging numbers in order is made easier by printing each number on a card so that the arrangement may be carried out physically. Similar practice is obtained through an incomplete set of numbers, written on separate cards, and with a limited range, say, 320 to 350. After arranging the numbers in order the pupils write down the missing numbers, also in order. Some children, especially boys, collect car numbers and here is another opportunity for ordered arrangement.

## COUNTING AND NUMBER RHYTHMS

A number square showing the numbers from 1 to 100 (Fig. 5) offers much scope for elementary practice. On the whole, it is better if it is made by the pupils than if duplicated copies are given to them. In the making, the number rhythms will become apparent. It can be used for counting in tens, starting at any number, such as 3, 13, 23, 33, . . . , etc. The pupils should be able to extend the counting beyond the square, . . . , 83, 93, 103, 113, etc. Counting backwards in tens can also be practised, 97, 87, 77, . . . .

The above practices help to show the decimal structure of our counting system. There are other practices which will help the child to acquire facility with the use of numbers, such as counting forwards in twos or threes, or any other regular interval. Any number may be a starting-point, for example, 11, 14, 17, 20, . . . . Counting backwards can also be practised. The less able children may need the number square to help them, counting the intervals and writing the answers as they go.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 11 | 21 | 31 | 41 | 51 | 61 | 71 | 81 | 91 |
| 2 | 12 | 22 | 32 | 42 | 52 | 62 | 72 | 82 | 92 |
| 3 | 13 | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 |
| 4 | 14 | 24 | 34 | 44 | 54 | 64 | 74 | 84 | 94 |
| 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 | 95 |
| 6 | 16 | 26 | 36 | 46 | 56 | 66 | 76 | 86 | 96 |
| 7 | 17 | 27 | 37 | 47 | 57 | 67 | 77 | 87 | 97 |
| 8 | 18 | 28 | 38 | 48 | 58 | 68 | 78 | 88 | 98 |
| 9 | 19 | 29 | 39 | 49 | 59 | 69 | 79 | 89 | 99 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

FIG. 5

For able pupils the sieve of Eratosthenes (Fig. 6) may be of interest; it will certainly be of value. It is a device for finding the prime numbers; it also illustrates factors and multiples. Using a number square the multiples of 2 are crossed off; that is, 4, 6, 8, 10, etc. The multiples of 3 are then cancelled. The first one, 6, has already been cancelled, so showing that 6 is a multiple of 2 and 3, or that 2 and 3 are both factors of 6. The multiples of 4 will be found already cancelled; they are all multiples of 2. The multiples of 5, 6, 7, 8, 9, 10, are dealt with in the same way. Since 10×10=100, multiples of numbers larger than 10 have already been cancelled, and the remaining numbers are the prime numbers less than 100.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 11 | 21 | 31 | 41 | 51 | 61 | 71 | 81 | 91 |
| 2 | ~~12~~ | ~~22~~ | ~~32~~ | ~~42~~ | ~~52~~ | ~~62~~ | ~~72~~ | ~~82~~ | ~~92~~ |
| 3 | 13 | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 |
| ~~4~~ | ~~14~~ | ~~24~~ | ~~34~~ | ~~44~~ | ~~54~~ | ~~64~~ | ~~74~~ | ~~84~~ | ~~94~~ |
| 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 | 95 |
| ~~6~~ | ~~16~~ | ~~26~~ | ~~36~~ | ~~46~~ | ~~56~~ | ~~66~~ | ~~76~~ | ~~86~~ | ~~96~~ |
| 7 | 17 | 27 | 37 | 47 | 57 | 67 | 77 | 87 | 97 |
| ~~8~~ | ~~18~~ | ~~28~~ | ~~38~~ | ~~48~~ | ~~58~~ | ~~68~~ | ~~78~~ | ~~88~~ | ~~98~~ |
| 9 | 19 | 29 | 39 | 49 | 59 | 69 | 79 | 89 | 99 |
| ~~10~~ | ~~20~~ | ~~30~~ | ~~40~~ | ~~50~~ | ~~60~~ | ~~70~~ | ~~80~~ | ~~90~~ | ~~100~~ |

FIG. 6

Number series (Chapter XVII) will also interest many, and at the same time give useful practice and provide an insight into number structures. Another source of number practice is the calendar. If 1 January is a Sunday, the other Sundays of January will be 8, 15, 22 and 29. The Mondays will be 2, 9, 16, 23, 30; and so on. (See also Chapter V, *Time*.)

An interesting five-minute diversion is to give the class a number, say 6, and discover how many facts about 6 they can write.

6 is half a dozen.
6th letter of the English alphabet is f.
6 ft.=1 fathom.
6 inches is half a foot.

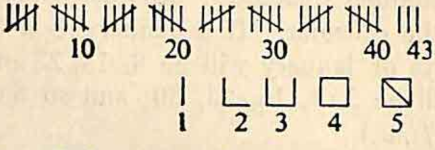
6th day of the week is Saturday.
A 6-sided figure is a hexagon.
6th month of the year is June.
6-day working week.
6 balls in an over (cricket).
6 is a maximum on dice and dominoes.

6 is a perfect number (i.e. its factors, 1, 2 and 3, add up to 6. The only other perfect numbers less than 1,000 are 28 and 496.)

When counting objects, as in a car census, some simple method of recording such as the ' gate ' method might be used. A stroke is made for each object up to the fourth, |, ||, |||, ||||. The fifth object is shown by a stroke passing through the other four, 卌 and the recording starts again. When finding the total score the groups are easily counted as fives; any remainder is added. Thus,

卌 卌 卌 卌 卌 卌 ||

is quickly counted as 5, 10, 15, 20, 25, 30, 32. There are many variations of this system:



At least one of these recording systems should be known and used.

## THE BASIC NUMBER COMBINATIONS

All calculations with numbers depend on a knowledge of the basic number combinations. The addition combinations for 2 are: $2+0$, $2+1$, $2+2$, $2+3$, ... $2+9$. If all the combinations are regarded separately, and we include zero combinations, there are 390 facts to be known. It is a mistake, however, to regard the combinations separately. For example, the number bonds,

$$5+3=8, \quad 3+5=8, \quad 8-3=5, \quad 8-5=3,$$

should be seen as a whole. They are different ways of expressing the relationship between 3, 5 and 8. Practice to emphasize this could be given from time to time, the children being asked to write all the possible equations for, say, the numbers 5, 7, 12.

## TABLES

Perhaps the most common complaint voiced by the school teacher is that his pupils do not know their tables. The tables referred to are the multiplication bonds but, from what has been stated in the previous paragraph, it is clear that these are only one part of the basic number bonds, all of which need practice. The multiplication bonds should be linked firmly with the division bonds:

$$7\times4=28, \quad 28\div4=7.$$
$$4\times7=28, \quad 28\div7=4.$$

With first-year children of poor ability it is advisable to re-teach the multiplication tables, and ensure that they possess meaning for them. The first step is to practise counting in twos, in threes, etc. Counting the class aloud after the register is called is a useful real situation. Other examples may be used, and the table can begin to appear in written form:

For 1 paisa I get 2 sweets.
,, 2 paise ,, 4 ,, $2\times2=4$.
,, 3 ,, ,, 6 ,, $3\times2=6$.
,, 4 ,, ,, 8 ,, $4\times2=8$, etc.

Similar tables can be built on 1 quart=2 pints, 1 rupee=2 half rupees, 1 bicycle has 2 wheels, etc. For very backward pupils, real objects or drawings may be needed. Only when the table of two has been seen in these real and concrete ways should any attempt be made to learn it by repeating it aloud.

It is desirable to concentrate on one table at a time. After the table of two has been discovered it can be used to answer questions such as:

*A toffee costs 2 paise. How much must I pay for:*

1. *3 toffees.* 2. *5 toffees.* 3. *6 toffees*, etc.

The table should also be used in reverse:

*How many toffees shall I get for:*

1. *4p.* 2. *8p.* 3. *14p.*, etc.

Repeating the tables aloud helps most children to memorize the number facts, but it must be stated emphatically that this is but one of many methods which should be employed. An intensive use of one table at a time, as shown in the examples above, is another method. Building and using a table square (Fig. 7) is also useful. As the pupils acquire a new table, it should be entered in their table square, which can then be used for simple mechanical practice. Once the child gains a little confidence in the use of a table, mental practice can help to speed up his response.

It is not necessary to learn the tables in numerical order. The table of two, followed by the table of three and the table of four, etc., is not the order of ease, nor is it an order of usefulness. There is no common agreement on what is the best order, but it is agreed that related tables should be learned together. This should not be carried too far, or it might mean that the first three tables learned are those of 2, 4, 8. One possible order is: 2, 4; 5, 10; 3, 6, 9, 12; 8; 11; 7.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 |
| 4 | 8 | 12 | | | | | | | | | |
| 5 | 10 | 15 | | | | | | | | | |
| 6 | 12 | 18 | | | | | | | | | |
| 7 | 14 | 21 | | | | | | | | | |
| 8 | 16 | 24 | | | | | | | | | |
| 9 | 18 | 27 | | | | | | | | | |
| 10 | 20 | 30 | | | | | | | | | |
| 11 | 22 | 33 | | | | | | | | | |
| 12 | 24 | 36 | | | | | | | | | |

FIG. 7. Table square, with table of two and table of three entered

Besides the basic number bonds, combinations of the type 32+6 are frequently encountered in column addition, and when carrying in multiplication. Practice with these bonds will therefore improve accuracy in computation.

## LONG MULTIPLICATION

Many pupils when they enter the secondary school will be found weak in multiplying by numbers greater than 12; some will not have encountered it at all. Whenever a weakness of this kind is found, it is advisable to re-teach from the earliest stages, rather than try to bolster up a half-assimilated technique by mechanical practice. The following method is recommended:

(i) Revise multiplication by 10. The very common formula, 'Add a nought' is strongly deprecated. In the example $23 \times 10$, we are finding the value of 23 tens. There will be no units so 0 is written in the units place.

(ii) Multiplication by 20, 30, etc. This extension should be regarded as multiplying by 2 tens, 3 tens, etc.

(iii) An example such as $41 \times 20$ is extended to $41 \times 23$:

$$\begin{array}{l} 41\times20=820 \\ 41\times\ \ 3=123 \\ \hline 41\times23=943 \end{array}$$

This layout should be used for a short period until the pupils acquire facility. The normal layout can then be substituted and the statements on the left-hand side omitted. With weaker children it is advisable to retain the 0 in the units column of the first partial product, instead of leaving a space as is often done. It will be noticed that the above teaching method leads to multiplying from the left side of the multiplier first.

Most children will be interested in the seventeenth-century multiplying device known as Napier's rods (Chapter XVI).

## LONG DIVISION

This is undoubtedly a difficult process and some children will never acquire a real grasp of it. It should be realized that there is no fundamental difference in method between 'long' and 'short' division. There is a difference of layout, and in 'short' division the multiples of the divisor are known from the multiplication tables.

One rather slow method which overcomes the early difficulties is first to write out the necessary multiplication table for the divisor to be used. If we are to divide by 24, we start by writing the table:

$2\times24=48$
$3\times24=72$
$4\times24=96$, etc.
. . . . .
to $9\times24=216$
and $10\times24=240$.

The last bond of this table is not required, but it is a useful check if the table is compiled by continuous addition instead of separate multiplications. The pupil then has a table to inspect at each stage of the work, and trial divisors are unnecessary. It is recognized, of course, that the time taken to write out the table is probably much greater than that taken to work the example. Such a method could not be generally contemplated. It is suggested, however, that during the first few weeks of practice in long division, the divisors used are restricted to about six and that the pupils have available the tables for these. In this preliminary work the divisors could be, say, 13, 15, 24, 33, 45, and 56. (21, 31, 41, etc., should be avoided since they are the easiest to use when the tables are dropped.) The time taken in writing out the tables is more than compensated when each table is used for several examples, and by the confidence gained in dealing with larger numbers.

The next stage is to omit the use of tables, although for many pupils it is advisable that the relevant parts of the required table are written out. The easy divisors 21, 31, 41, etc., are used and trial divisors are introduced. In the example $21\overline{)672}$, to find the answer to $67 \div 21$, the child uses $6 \div 2$ to give the first figure of the quotient. He then goes on to write $21 \times 3 = 63$. Inspection shows that this is satisfactory and it is entered in the working.

## REMAINDERS

Many children are doubtful how to treat remainders in division. Should the answer to $7 \div 3$ be written as 2, r. 1 or $2\frac{1}{3}$? Both are correct, which is to be preferred? The difficulty generally arises through large quantities of mechanical practice unrelated to real problems. If the division which occurs in any real situation is examined, the way in which the remainder should be treated becomes clear. The problem of dividing a length 7 cm into 3 equal parts is satisfied by 3 lengths, each $2\frac{1}{3}$ cm. If 7 children are divided into 3 groups by a similar method, the result would undoubtedly produce some dissatisfied parents.

## VULGAR (COMMON) FRACTIONS

Children become acquainted with terms such as half, quarter, from everyday experience; such quantities are in common use. The range of fractions used in school should be strictly limited, and 'awkward' fractions such as $\frac{1}{23}$, $\frac{9}{29}$, etc., should be avoided. Such quantities are better expressed as decimal fractions (to 3 or 4 decimal places if necessary), and are easier to use in that form.

Primary schools teach some fraction work, and so at higher levels the treatment of addition and subtraction of fractions may be largely revisionary. For some pupils material of the simplest kind may be necessary to help their understanding and use of fractions. The most suitable form is probably paper strips which can be folded to give halves, quarters, and eighths (Fig. 8).

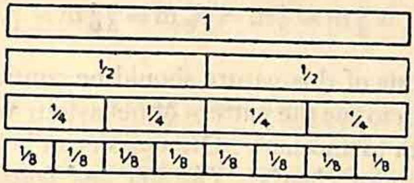


Fig. 8

Strips showing thirds can be folded to give sixths and twelfths, so providing an excellent range of fractions for use. It is not easy to fold a strip into three parts with accuracy. The simplest method is to cut the strips 15 cm long and mark off into 5 cm divisions. The children then fold these to show that the divisions are thirds.

Another approach to fraction work, suitable for most children, is through measurement in centimetres and millimetres.

There is no general agreement as to when the terms *numerator* and *denominator* should be introduced. Many teachers use *top* and *bottom* to name the parts. It is useful, however, to indicate the functions of the two parts, and for this purpose some teachers prefer to use *number* and *name* for *numerator* and *denominator* respectively. The latter states the name of the fractional parts; the former states the number of such parts. The terms *numerator* and *denominator* may slowly replace the simpler terms.

When the nature of a vulgar fraction is recognized, the equivalence of fractions must be established. From the strips it can quickly be seen that

$$\tfrac{1}{2}=\tfrac{2}{4}=\tfrac{3}{6}=\tfrac{4}{8}=\tfrac{6}{12}.$$

If a length of $\frac{1}{2}$ m is marked out and then measured using different fractional parts, it will be seen that

$$\tfrac{1}{2}\,\text{m}=\tfrac{2}{4}\,\text{m}=\tfrac{4}{8}\,\text{m}=\tfrac{5}{10}\,\text{m}=\tfrac{25}{50}\,\text{m}=\tfrac{50}{100}\,\text{m}.$$

Demonstrations of this nature should be continued until the children begin to see the pattern of behaviour when changing from one form to another. Discussion can help to establish the pattern more clearly, '*The top and bottom parts of a fraction must both be multiplied by the same number*'. The pupils should then be asked to convert fractions using the rule they have found, afterwards checking their results by the strips, or through measurement, or by some other practical aid.

A sound knowledge of equivalence is essential before proceeding to addition and subtraction of fractions. In most cases, when adding or subtracting, the children should be able to see the common denominator needed. If they cannot do so, the method of multiplying the largest denominator by 2, 3, 4, etc., and testing each time, is much to be preferred to the factor method of finding the L.C.M. (The common denominator for $\frac{3}{8}+\frac{1}{4}+\frac{5}{6}$ is found by trying in turn, 8, 8×2, 8×3.)

In the early work the use of a single denominator should be avoided. For example, $\frac{1}{2}+\frac{3}{8}$ should be rewritten as $\frac{4}{8}+\frac{3}{8}$, not $\frac{4+3}{8}$. Many children find difficulty in identifying the parts of the latter with the separate fractions of the original statement.

In addition and subtraction of mixed numbers, the whole numbers should be treated first, then the fractional parts. The difficulty which sometimes arises in subtraction is best dealt with in the following manner:

$$\begin{aligned} 4\tfrac{1}{2}-1\tfrac{3}{4} &= 3\tfrac{1}{2}-\tfrac{3}{4} \\ &= 3\tfrac{2}{4}-\tfrac{3}{4} \\ &= 2\tfrac{4}{4}+\tfrac{2}{4}-\tfrac{3}{4} \\ &= 2\tfrac{6}{4}-\tfrac{3}{4} \\ &= 2\tfrac{3}{4}. \end{aligned}$$

While there is a strong and clear case for including addition and subtraction of fractions in the school syllabus, multiplication and division by fractions present another picture. To give meaning to $\frac{3}{8}\times\frac{1}{4}$ is not easy, nor is it an operation in common use. In practice the operation which does occur is $\frac{3}{8}$ of $\frac{1}{4}$, and it is this aspect which should be treated. The mechanical operations of multiplication and division are generally found easy, but the difficulty of giving meaning to them suggests that their introduction should be delayed or, in some cases, omitted altogether. In any case it is inadvisable to teach all four rules in fractions at the same time. It is important that the processes of addition and subtraction are thoroughly absorbed before the others are tackled.

Multiplication of a fraction by a whole number offers little difficulty, and practical cases are readily found. The cost of 5 notebooks at Re $\frac{3}{4}$ each is found by the operation $\frac{3}{4}\times 5$. Most difficulties at this stage are overcome by an insistence on the functions of the numerator and denominator. The latter tells us the name of the fractional parts we are dealing with; the former tells us the number of parts, and is obviously the quantity to be multiplied.

Area diagrams can be used to show a fraction of a fraction (Fig. 9). The rule for multiplying fractions should not be given until the pupils are well on the way to discovering it for themselves. They will be helped to do so if they are encouraged to look at the numbers in each finished example to see if there is any common pattern of behaviour.

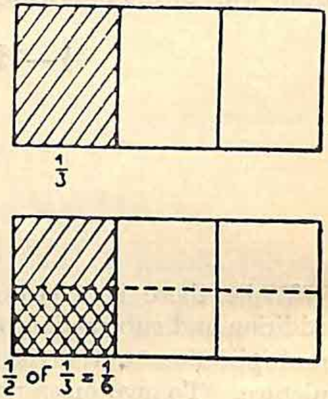


FIG. 9

It is recommended that cancelling be delayed until the teacher is sure that all fractional operations are understood and carried out with ease. Cancelling should be regarded as a simplifying process applied before the multiplication, and thus saving time. In the example

$$\frac{2}{5}\times\frac{3}{8}=\frac{6}{40}=\frac{3}{20},$$

it can be pointed out that in the top line we have first multiplied by 2, and then in the reduction we have divided by 2. This is a waste of time, and the reduction could have been done at the beginning of the work. Some teachers prefer to start by writing:

$$\frac{2}{5}\times\frac{3}{8}=\frac{2\times 3}{5\times 8},$$

then cancelling: $$\frac{\overset{1}{\cancel{2}}\times 3}{5\times\underset{4}{\cancel{8}}}=\frac{3}{20}.$$

A number should always be cancelled by a single stroke, thus $\cancel{12}$, not $\cancel{1}\cancel{2}$. In bad cases the latter can lead to errors such as:

$$\frac{\cancel{4}}{9}\times\frac{5}{1\cancel{4}}.$$

Division of fractions can be illustrated by finding how many half-cm there are in $2\frac{1}{2}$ cm,

$$\text{i.e. } 2\tfrac{1}{2}\div\tfrac{1}{2}.$$

It is advisable to treat such elementary examples as 'How many halves in 5 halves?' rather than to use the normal division method $(\frac{5}{2}\times\frac{2}{1})$. Examples such as $1\frac{1}{2}\div\frac{1}{6}$ can be treated in a similar manner

$$\begin{aligned}\tfrac{3}{2}\div\tfrac{1}{6}&=\tfrac{9}{6}\div\tfrac{1}{6}\\&=9.\end{aligned}$$

The fractions are reduced to the same denominator (or, shall we say, denomination viz., the *one-sixths*).

This is the method which is most commonly used in mental calculation. Such questions as 'How many half-cm in $5\frac{1}{2}$ cm?' $(5\frac{1}{2}\div\frac{1}{2})$, or 'How many quarter-cm in $8\frac{1}{2}$ cm?' $(8\frac{1}{2}\div\frac{1}{4})$ are frequently done mentally, and the above method is invariably used.

For more difficult calculations the normal method must be used. It is probably best introduced by a series of examples such as:

1. (*a*) $\frac{3}{2}\div\frac{3}{8}$ (*b*) $\frac{3}{2}\times\frac{8}{3}$
2. (*a*) $\frac{5}{4}\div\frac{5}{12}$ (*b*) $\frac{5}{4}\times\frac{12}{5}$
3. (*a*) $\frac{6}{5}\div\frac{3}{10}$ (*b*) $\frac{6}{5}\times\frac{10}{3}$, etc.

Some children find it difficult to recall whether the divisor or dividend has to be inverted. (Does $\frac{3}{8}\div\frac{1}{4}$ become $\frac{3}{8}\times\frac{4}{1}$ or $\frac{8}{3}\times\frac{1}{4}$?) In such cases the difficulty can generally be overcome by emphasizing that the operation being carried out is division by $\frac{1}{4}$. This links the division sign more firmly with the divisor.

## DECIMAL FRACTIONS

After the Hindu conception of the modern counting system about A.D. 600, it was a thousand years before an extension of the system to include fractional parts was described by Stevin in 1585. Many children are acquainted with 0·1 as $\frac{1}{10}$, for example, in race times, 9·4 seconds. It is important, however, that they realize this form is a part of the normal counting system. In this way they see that the ordinary methods of calculating apply to decimal fractions. The particular variations found in decimal fraction calculations are solely to decide the position of the decimal point. They do not affect the actual operations of addition, etc., in any way.

Position value is an important aspect of our counting system; it is therefore necessary to revise it before starting work on decimal fractions. The introduction can be through operating on numbers written in column form. After some preliminary practice in writing numbers in this manner, a number such as 53 is multiplied repeatedly by 10.

| LAKHS | | THOUSANDS | | | | |
|---|---|---|---|---|---|---|
| *Tens* | *Units* | *Tens* | *Units* | *Hundreds* | *Tens* | *Units* |
| | | | | | 5 | 3 |
| | | | | 5 | 3 | 0 |
| | | | 5 | 3 | 0 | 0 |
| | | 5 | 3 | 0 | 0 | 0 |
| | 5 | 3 | 0 | 0 | 0 | 0 |
| 5 | 3 | 0 | 0 | 0 | 0 | 0 |

This operation is also carried out with other numbers until the class see quite clearly the movement of the figures across the columns. This movement must not be given as a rule, but attention can be drawn to it by a question such as: 'What happens to the figures every time we multiply by 10?' In the next series, multiplying by 10 should be continued until the left-hand figure moves outside the columns.

|  | LAKHS |  | THOUSANDS |  |  |  |  |
|---|---|---|---|---|---|---|---|
|  | *Tens* | *Units* | *Tens* | *Units* | *Hundreds* | *Tens* | *Units* |
|  | 4 | 6 | 0 | 0 | 0 | 0 | 0 |
| 4 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |

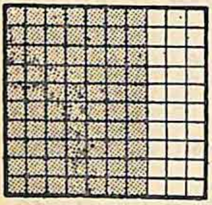
A new column is drawn for the figure, and its heading, *Crores*, inserted.

The same procedure is then carried out for division by 10. When the class are quite clear about the movement of the figures for division the process is repeated until the right-hand figure moves outside the columns.

| LAKHS |  | THOUSANDS |  |  |  |  |  |
|---|---|---|---|---|---|---|---|
| *Tens* | *Units* | *Tens* | *Units* | *Hundreds* | *Tens* | *Units* |  |
| 7 | 3 | 0 | 0 | 0 | 0 | 0 |  |
|  | 7 | 3 | 0 | 0 | 0 | 0 |  |
|  |  | 7 | 3 | 0 | 0 | 0 |  |
|  |  |  | 7 | 3 | 0 | 0 |  |
|  |  |  |  | 7 | 3 | 0 |  |
|  |  |  |  |  | 7 | 3 |  |
|  |  |  |  |  |  | 7 | 3 |

A new column is drawn for it and its heading, *Tenths*, is found. This may be done by inspection of the column headings. As we read from left to right the column values decrease by steps of tenths; each column heading is $\frac{1}{10}$th of its neighbour on the left. The new heading will therefore be $1 \div 10 = \frac{1}{10}$. Alternatively, the last calculation, $73 \div 10$, can be carried out to give an answer of $7\frac{3}{10}$. The 3 in the new column stands for 3 tenths, and the column is therefore a *Tenths* column. On the blackboard the new column can be shown in coloured chalk.

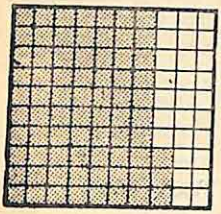
Practice should then be given in writing numbers such as $23\frac{7}{10}$ in column form and in changing from column form to mixed numbers.



0.7 of the area

FIG. 10

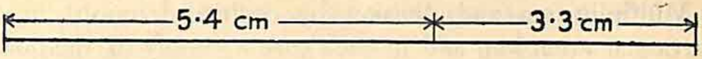
One way of showing the need for the decimal point is to write some numbers on the blackboard in columns. These should include numbers with tenths. If the lines which form the columns are then rubbed out, leaving only the numbers, it becomes obvious that a method of showing which is the units figure is necessary.



0.73 of the area

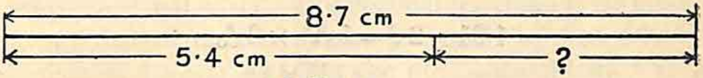
FIG. 11

Decimal fractions should then be used and illustrated in a variety of ways. An area of 1 centimetre square on graph paper can be shaded to show 0·1 sq. cm, 0·2 sq. cm, etc. (Fig. 10). This diagrammatic representation is also useful for illustrating the second decimal place when it is introduced later on (Fig. 11). From now on lengths should frequently be written using the decimal notation. Simple addition and subtraction of decimals can be illustrated by drawing and measuring lines (Figs. 12, 13).



$$\begin{array}{r} 5{\cdot}4\text{ cm} \\ +\,3{\cdot}3\text{ cm} \\ \hline 8{\cdot}7\text{ cm} \end{array}$$

FIG. 12



$$\begin{array}{r} 8{\cdot}7\text{ cm} \\ -\,5{\cdot}4\text{ cm} \\ \hline 3{\cdot}3\text{ cm} \end{array}$$

FIG. 13

Some time should elapse before the second decimal place is introduced. It may be taught by a method using figures in columns, similar to that employed for the introduction of the first decimal place; or it may be approached through measurement using the decimetre as a unit. The advantage in using this rather unusual unit is that it is possible to measure to the nearest hundredth of the unit accurately.

The second decimal place arises naturally from this situation if the pupils imagine, say, each $\frac{1}{10}$ dm divided into 10 parts. One of these parts will be $\frac{1}{100}$ decimetre, or 0·01 decimetre (1 millimetre).

The only new difficulty which arises in addition and subtraction of decimals lies in their setting out. Examples written on the blackboard for the class to work should be arranged horizontally, such as:

$$13{\cdot}21 + 0{\cdot}248 + 536{\cdot}7 + 14{\cdot}925$$

or

$$53{\cdot}2 - 7{\cdot}84,$$

so that the pupils must first set them out correctly.

Multiplication and division by decimal fractions need particular attention, and in each case a variety of methods is possible. The simplest method of multiplication is that of 'counting the places', but it is often criticized because it is learned, without understanding, as a rule of thumb. The following introduction leads easily and naturally to 'counting the places' and provides the reason for it.

$$\begin{aligned} 1{\cdot}31 \times 2{\cdot}3 &= 1\tfrac{31}{100} \times 2\tfrac{3}{10} \\ &= \tfrac{131}{100} \times \tfrac{23}{10} \\ &= \tfrac{3013}{1000} \\ &= 3{\cdot}013 \end{aligned} \qquad \begin{array}{r} 131 \\ 23 \\ \hline 393 \\ 2620 \\ \hline 3013 \end{array}$$

A number of similar examples should be worked by the class, using numbers which lead to answers with 2, 3, or 4 decimal places. The method should then be analysed. It will be noticed that we take the two numbers and multiply them together, disregarding the decimal point. The next step is to fix the position of the decimal point. An inspection of the examples done and their answers should lead to the desired rule. Some attempt should be made to express a reason for it. In the example, $1{\cdot}31 \times 2{\cdot}3$, we are multiplying hundredths by tenths. The answer will therefore be in thousandths and there will be 3 decimal places.

When dividing by a decimal fraction, the simplest methods are those which change the divisor to a whole number or to standard form.† The dividend is compensated as shown in the following example:

$52{\cdot}29 \div 16{\cdot}4$

becomes $522{\cdot}9 \div 164$ (Divisor a whole number)

or $5{\cdot}229 \div 1{\cdot}64$ (Divisor in standard form)

---

† Numbers between 1 and 10 are said to be in standard form; they have one figure before the decimal point.

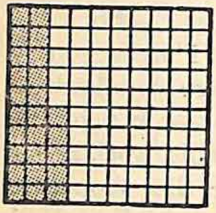
The former is the more common, and is probably more suitable for pupils.

It often happens that when a new notion or process is introduced, it leads to other ideas which are not evident at first. The multiplication of vulgar fractions extends the meaning of multiplication; and in decimal fractions the non-terminating decimal is encountered. (The fraction $\frac{7}{31}$ cannot be stated exactly as a decimal for there is no limit to the number of decimal places to which we can take the quotient of $7 \div 31$.) At first the pupils should work as far as the second or third decimal place. Later they should be taught how to give answers *correct* to the second or third decimal place. This entails working to one more place than is needed. For example, the answer 30·137 is written 30·14 correct to two decimal places.

At a later stage the value of writing an answer to so many significant figures might be shown. The numbers 98·18 and 0·02 are both written to two decimal places but the percentage accuracy is much greater in the first than in the second. The possible error in each case is $\pm 0{\cdot}005$, which is 25 per cent of the second number, but only about 0·005 per cent of the first. If the numbers 3,586·258 and 11·732 are both written to 3 significant figures (3,590 and 11·7) they are both written to the same degree of percentage accuracy. (It should be noted that in a number such as 0·002856, the noughts before the 2 are not counted as significant figures. To three significant figures the number is 0·00286.)

## PERCENTAGES

A percentage can be regarded as a fraction with denominator 100, or as indicating so many parts per 100 parts. (e.g. 75 per cent of the population; and a premium of Rs 2 per cent.) The latter was the original meaning of percentage, but has since fallen out of use in most countries. It is advantageous if pupils can regard percentages in both ways. Examples are sometimes made easier if the percentages are thought of as so many parts out of 100. The link between a percentage, 15 per cent, and its decimal equivalent, 0·15, should be clear, especially since quantities are stated sometimes in one form, sometimes in the other. The decimal form is more common now than it was some years ago. The link is usefully illustrated by the 100 small squares in a unit square on graph paper (Fig. 14).



0·25 or 25 %
of the area
FIG. 14

Percentages are generally used to indicate parts of a whole, and children should be shown their use in comparing parts of different wholes. For example, 2,000 people using the public library in a town of 5,000 population is a greater fraction than 3,000 people using the public library in a town of 10,000.

The pupil should be able to understand what is meant by, say, 80 per cent of the class; and know how to find a percentage of a given quantity:

$$\text{e.g. } 80 \text{ per cent of } 40 = \tfrac{80}{100} \times 40 = 32.$$

The child should also be able to state a part as a percentage,

e.g. 32 children out of 40.

$$\text{Fractional part} = \tfrac{32}{40}.$$

$$\text{Percentage} = \tfrac{32}{40} \times 100 \text{ per cent} = 80 \text{ per cent.}$$

(The method of changing a fraction to a percentage is a simple one, but it is surprising how few adults recognize the reason for it. When we change any quantity to a percentage we are changing it to hundredths. To find the number of hundredths, we must multiply by 100.

i.e. In 2 there are $2\times100$ hundredths,
,, 6 ,, ,, $6\times100$ ,,
,, $\frac{1}{2}$ ,, ,, $\frac{1}{2}\times100$ ,, etc.)

There are many particular cases where percentages are commonly used, e.g. 10 per cent discount, 5 per cent experimental error, $2\frac{1}{2}$ per cent interest, 100 per cent attendance, etc. Most of these can be tackled at appropriate occasions in other work. The problems on profit and loss, etc., found in many textbooks have little value apart from testing the pupils' ability to manipulate percentages.

## RATIO AND PROPORTION

Despite the fact that both these related mathematical notions are encountered frequently in everyday life, many children find difficulty when they are treated formally. A ratio is a comparison between two like quantities and the young child who says '*Twice as big*', or '*Ten times as long*', is using ratio. He also makes use of proportion in situations such as, '*Twice as many sweets, twice as much to pay*,' or '*12 p. fare for one person, 36 p. for 3 people*'. As long as the numbers involved are simply related, children can deal with problems of this type, using their experience as a guide. Work of this kind is largely intuitive, however, and slightly more difficult number relationships often produce complete failure. The ratio of 6 cm to 2 cm is easily reduced to '3 times as big'. The ratio of 5 cm to 2 cm provides much greater difficulty, for the child is almost certainly unaware of the first step he has taken in the simple case. If he could see the first ratio as $\frac{6}{2}$ (or 6:2), there should be no difficulty whatsoever in seeing the second ratio as $\frac{5}{2}$ (or 5:2).

For this reason it is advisable to do some very simple work emphasizing ratio; for example, measuring lines 2 cm and 3 cm in length and writing the ratio of their lengths as 2:3. The child readily reduces a ratio of 6:2 to 3:1; he should also be able to reduce, say, 10:4 to its simpler form, 5:2, etc.

For most pupils the *unitary method* is probably the best method of solving proportion problems. For example:

*If 7 chairs cost Rs 77, what is the cost of 4 chairs?*

*7 chairs cost Rs 77.*

$$1 \text{ chair costs } \frac{Rs\ 77}{7} = Rs\ 11.$$

$$4 \text{ chairs cost } Rs\ 11 \times 4 = Rs\ 44.$$

It is generally advised that the fraction in the second line of the solution $\left(\frac{Rs\,77}{7}\right)$ should not be evaluated, but that all working be done in the final stage. This is undoubtedly a saving of work in many examples, but some pupils find the statement *1 chair costs Rs 11* far easier to comprehend than *1 chair costs* $\frac{Rs\ 77}{7}$.

The order of the wording in the first line of unitary examples needs some treatment. In the solution to the problem:

*A car travels 120 kilometres in 5 hours. How far will it travel in 6 hours at the same rate?*

it is likely that many pupils will write for their first line:

*A car travels 120 kilometres in 5 hours,* copying this statement from the question. This will not lead easily to a correct solution. The pupils should first decide on the units of the answer, and these units must appear at the end of the first line. In this case, the answer will be in kilometres, and the required statement is:

*In 5 hours a car travels 120 kilometres.*

One objection to the unitary method is that the second statement is often absurd, as in the example,

*30 hens produce an average of 135 eggs per week. How many hens are needed to produce an average of 450 eggs per week?*

The second line will read

*1 egg is produced by* $\frac{30}{135}$ *hens.*

Able pupils should be encouraged to omit this second line of the solution, once the method is thoroughly grasped. The solution then becomes:

*135 eggs are produced by 30 hens*
*450 ,, ,, ,, ,,* $30 \times \frac{450}{135}$ *hens.*

This closely approximates to the method of solution known as the *ratio method*[†]. Notice how the fraction $\frac{450}{135}$ is obtained. In solutions of this kind, such fractions are referred to as *Multiplying Fractions*. If instead of 450 eggs we were to consider 320 eggs, the multiplying fraction would be $\frac{320}{135}$. It is undoubtedly the best approach for pupils.

One advantage of the unitary method (and its shortened form) is that a very wide range of examples can be solved by it. For example, all questions on Simple Interest can be answered by using the unitary method. The formula $I = \frac{Pnr}{100}$ only serves to calculate interest ($I$). If the principal ($P$), rate per cent (r), or time in years (n) are required the pupil must either be able to transpose the subject of a formula to obtain, for example, $r = \frac{100\,I}{Pn}$, or he must memorize four formulae. A better method is to emphasize the meaning of a rate per cent, and to write an appropriate statement.

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[†] See Mathematical Association report *The Teaching of Arithmetic* (Bell), page 45.

e.g. *At 2½ per cent the interest on Rs 100 for 1 year is Rs 2·50.*

The solution then becomes an instance of a general mathematical method instead of an isolated piece of work as is the case when formulae are used.

The wording of the statement is altered according to the quantity it is desired to find. If it is desired to find the principal, the first statement becomes:

*Interest of Rs 2·50 is obtained in 1 year on Rs 100.*

## LOGARITHMS

The decision to include logarithms in the syllabus depends chiefly on the use which can be made of them. Logarithms only acquire value through usage: from usage springs confidence, and confidence leads to further use. If the syllabus of the final year does not include much work in which logarithms can be used, then their value is doubtful. On the other hand the ease of calculation when using logarithms makes them a source of joy to the pupil who is wearied of computation.

Powers of numbers should be treated before logarithms are taught; squares and square roots can enter the syllabus of the first or second years, and an extension may be made later to cubes and higher powers. It is advisable to describe the index number in, say, $5^4$, as telling us how many fives are multiplied together. The economy of the index notation in dealing with very large numbers should also be demonstrated.

Some teachers like to show how the logarithm tables may be used to ease multiplication and division of numbers, without any attempt to explain the basic mathematical ideas involved. This certainly gives a rapid introduction to the use of logarithms, and will appeal to some pupils. If this approach is used, then some explanation should be given at a later stage.

The index laws upon which the use of logarithms is based are best demonstrated using 2 as an example.

$$2^2 = 2\times2 = 4$$
$$2^3 = 2\times2\times2 = 8$$
$$2^4 = 2\times2\times2\times2 = 16, \text{ etc.}$$

The pupils should write this table as far as $2^{12}$. Examples of the following type are then worked:

$$\begin{aligned} 16\times64 &= 2^4\times2^6 \quad \text{(from the table)} \\ &= 2\times2\times2\times2 \ \times \ 2\times2\times2\times2\times2\times2 \\ &= 2^{10} \\ &= 1024 \quad \text{(from the table).} \end{aligned}$$

After working a few of these examples the brighter pupils will realize that the indices are being added, and the second line of working will be omitted.

Division is treated in a similar manner; it will be seen that in multiplication the indices are added, while in division the indices are subtracted. To emphasize these index laws the work may be repeated using the table of three, i.e.

$$3^2 = 9$$
$$3^3 = 27$$
$$3^4 = 81, \text{ etc.}$$

The next step is to extend the indices used to include 1 and 0. This is best done by inspecting the table of two:

$$64 = 2^6$$
$$32 = 2^5$$
$$16 = 2^4$$
$$8 = 2^3$$
$$4 = 2^2.$$

The left-hand side can be seen as a series of numbers, each a half of the preceding one. The next two numbers are, therefore, 2, 1. Inspecting the right-hand side shows the indices decreasing by 1 each time, so that the next indices will be 1, 0. Thus

$$2 = 2^1$$
$$1 = 2^0.$$

It should be realized that the simple meaning given so far to a power is now being extended to cover a much wider field, and one which is by no means easy to define in simple terms. On the whole it is better not to attempt a definition of $2^0$ and similar expressions; it is sufficient to show that they have a real value. A number of examples of the following kind should be worked on the blackboard:

(*a*) $32 \div 16 = 2$

Using indices: $2^5 \div 2^4 = 2^{5-4}$

$= 2^1$,

i.e. $2^1 = 2.$

(*b*) $16 \div 16 = 1$

Using indices: $2^4 \div 2^4 = 2^{4-4}$

$$= 2^0,$$

i.e. $2^0 = 1.$

The work should be repeated using the tables of three, four, etc., so approaching the general rule: $x^1 = x$, $x^0 = 1$. It will not, of course, be stated in this algebraic form.

The next step is to introduce fractional indices. The table of two is once more written in the form:

$$\begin{aligned} 1 &= 2^0 \\ 2 &= 2^1 \\ 4 &= 2^2 \\ 8 &= 2^3, \text{ etc.} \end{aligned}$$

An obvious defect of this table for calculation is that it does not contain all numbers. It will be seen that 3, lying between 2 and 4, is also a value between $2^1$ and $2^2$. It is, in fact, $2^{1.6}$. This use of an index other than a whole number is yet another extension to the meaning of an index. Again, no explanation should be attempted, but the fact that numbers with fractional indices can have real values may be demonstrated in the following manner:

$$\begin{aligned} 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}} \\ &= 2^1, \end{aligned}$$

i.e., $2^{\frac{1}{2}}$ when multiplied by itself is equal to 2.

$$\begin{aligned} \therefore 2^{\frac{1}{2}} &= \sqrt{2} \\ &= 1{\cdot}414. \end{aligned}$$

The square roots of other numbers may be used in a similar manner; cube roots, etc., may also be used.

The tables of common logarithms to base 10 may then be introduced and some practice given in using them to find logarithms of numbers. In the early stages the mean-difference columns should be ignored. 3-figure tables without mean-difference columns should be used if available. The numbers must then be restricted to no more than three figures. The following layout of examples is recommended:

$$\begin{array}{lr} 2{\cdot}73 \times 1{\cdot}06 = 10^{\cdot 436} \times 10^{\cdot 025} & \cdot 436 \\ \qquad\qquad\quad = 10^{\cdot 461} & \underline{\cdot 025} \\ \qquad\qquad\quad = 2{\cdot}89. & \cdot 461 \end{array}$$

The showing of the powers of 10 helps understanding at this stage, while the vertical addition of powers shown on the right-hand side avoids many errors.

At first, the table of logarithms should be used to find the number 2·89 from the logarithm ·461. The body of the table is searched to find ·461, or the nearest value to it; the number is then read off. This using of the logarithm table in reverse helps to demonstrate the function of the anti-logarithm table, which can soon be introduced.

Characteristics can be avoided in the early work by the following method:

$$\begin{aligned} 39{\cdot}2 \times 2{,}310 &= 3{\cdot}92 \times 10 \times 2{\cdot}31 \times 1{,}000 \\ &= 3{\cdot}92 \times 2{\cdot}31 \times 10{,}000. \end{aligned}$$

Logarithms are then used to find the value of $3{\cdot}92 \times 2{\cdot}31$, and the answer is multiplied by 10,000. This method is fundamentally the same as that on which characteristics are based and provides useful practice before characteristics are introduced. Where it is felt that the use of negative characteristics is beyond the ability of a class this method may be employed for numbers less than 1.

4-figure tables with their greater accuracy should be used as soon as the pupils acquire some competence in using logarithms. When using 4-figure tables answers should generally be given to 3 figures, as the fourth figure is not always accurate. Answers are probably correct to 4 figures provided:

(*a*) no logarithms have been multiplied to find powers of numbers,

(*b*) not more than 6 logarithms have been used in the calculation,

(*c*) the first two figures of the answer are less than 44.

So far the numbers dealt with by logarithms have been in standard form (page 58). We can find the logarithm of a number greater than 10 in the following manner:

$$\begin{aligned} 53{\cdot}85 &= 5{\cdot}385 \times 10 \\ &= 10^{0.7312} \times 10^{1} \\ &= 10^{1.7312} \end{aligned}$$

Similarly

$$\begin{aligned} 538{\cdot}5 &= 10^{2.7312} \\ 5{,}385 &= 10^{3.7312}, \text{ etc.} \end{aligned}$$

The part of the logarithm after the decimal point (·7312) remains the same; the number in front, the characteristic, changes. It indicates the position of the decimal point in the answer. If the characteristic is 0 the answer is in standard form. The value of the characteristic tells us how many places the decimal point in the answer is moved from its standard form position.

e.g. $10^{0.3156} = 2{\cdot}068$ (*Characteristic 0, answer in standard form.*)

$10^{1.3156} = \overset{\frown}{20}{\cdot}68$ (*Decimal point moved 1 place.*)

$10^{2.3156} = \overset{\frown}{20}\overset{\frown}{6}{\cdot}8$ ( ,, ,, *2 places.*)

$10^{3.3156} = \overset{\frown}{20}\overset{\frown}{6}\overset{\frown}{8}{\cdot}$ ( ,, ,, *3 places.*)

To find the value of $10^{2.5312}$ we first find the antilogarithm corresponding to ·5312. The characteristic, 2, tells us that the decimal point is moved 2 places to the right from its standard form position.

$$10^{2.5312} = \overset{\frown}{3}\overset{\frown}{3}9{\cdot}8.$$

This is by far the best method of finding characteristics, and of placing the decimal point in the answer; the same rule applies in each case. It also applies to negative characteristics. When teaching characteristics from a blackboard it is useful to write the characteristic in coloured chalk in the early stages of the work.

By this time the following layout of examples should be adopted:

$$13{\cdot}73 \times 5{\cdot}271 \times 683{\cdot}1$$
$$= 49{,}400$$

| *No.* | *Log* |
|---|---|
| 13·73 | 1·1377 |
| 5·271 | 0·7219 |
| 683·1 | 2·8345 |
| 49,440 | 4·6941 |

If negative characteristics are to be used their introduction must be preceded by work on directed numbers, including addition and subtraction of directed numbers (see page 74.) The fact that characteristics are directed numbers should be emphasized; a *positive* characteristic indicates *multiplication* by 10, 100, etc.; a *negative* characteristic indicates *division* by 10, 100, etc.

## THE SLIDE RULE

The use of the slide rule is not included at present in the High School or the Higher Secondary School syllabus. It is hoped, however, that this very useful device will find a place in the syllabus sooner or later.

Work on the slide rule should certainly accompany work on logarithms; it can illustrate and clarify several of the steps taken. Many children who would find difficulty in logarithms can enjoy using a slide rule. The ease with which multiplication and division can be carried out makes a strong appeal to most pupils.

It can be introduced quite early in the work on decimals, using linear scales for addition and subtraction. Two strips of 1 cm and 0·1 cm graph paper are marked in centimetres as shown in Fig. 15. It is advisable to label them A and B.

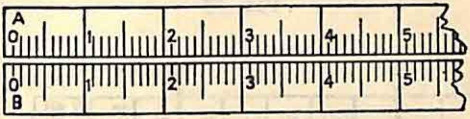

FIG. 15

To find the value of 1·4+2·9, the end of the B scale is placed against 1·4 on the A scale. The answer is found on the A scale opposite 2·9 of the B scale (Fig. 16). It will be seen that the answer is found by adding lengths of 1·4 cm and 2·9 cm. The use of the scales for subtraction is illustrated in Fig. 17.

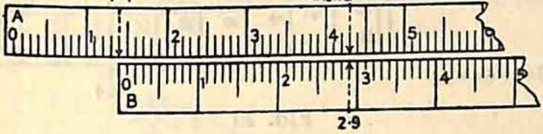

FIG. 16. $1·4+2·9=4·3$

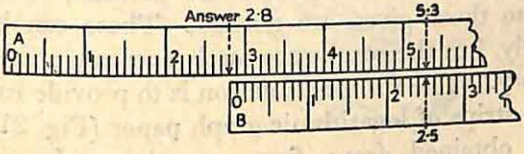

FIG. 17. $5·3-2·5=2·8$

In Fig. 18 logarithmic (or multiplying) scales are substituted for the linear scales. Two strips of plain paper, divided accurately into $\frac{1}{2}$ cm intervals, are marked with the table of two. Figs. 19 and 20 show the scales being used for multiplication and division.

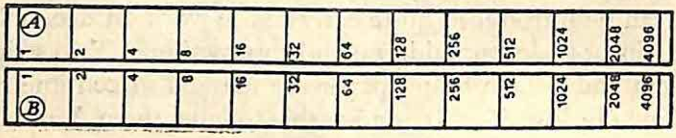

FIG. 18

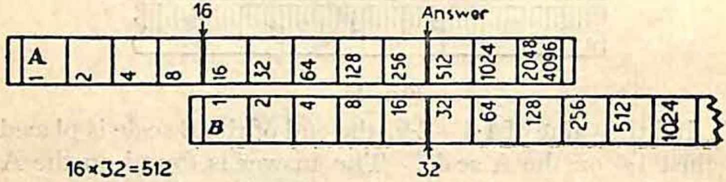

FIG. 19

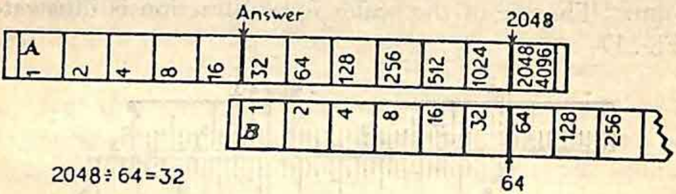

FIG. 20

The limitations of these scales are obvious; values intermediate to those given are needed. These can be found graphically, but the results are rarely accurate enough to be satisfactory. The simplest solution is to provide each pupil with two strips of logarithmic graph paper (Fig. 21). This may be obtained from firms supplying drawing-office equipment, etc., and a few sheets would provide enough strips for a class. The position of 1, 2, 4, 8, etc., on the logarithmic graph paper should first be noted. As in the previous simple scale they will be found to be equally spaced. The new scale will thus be seen to be similar to the original scale except that we now have a full range of values.

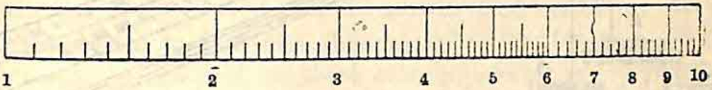

FIG. 21

It is important that the pupils should realize that the range of numbers on the slide rule is an unrestricted one. One operation provides the answers to $0{\cdot}71\times0{\cdot}23$, $7{\cdot}1\times2{\cdot}3$, $71\times23$, $0{\cdot}071\times230$, etc. This work is of particular benefit before the introduction of characteristics in logarithms. Rough answers are useful as a check as well as giving the position of the decimal point. For example, if we wish to find the value of $2{\cdot}3\times560$, the answer is, very roughly, $2\times600 = 1200$. Such checks can, of course, be used in a wide range of work.

Figs. 22 and 23 show two simple constructions for making slide rules in wood and in card.

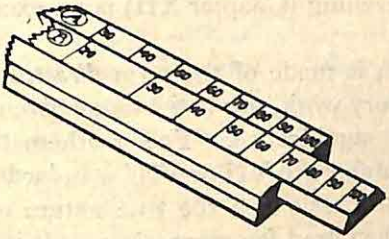

FIG. 22. Slide rule made of wood

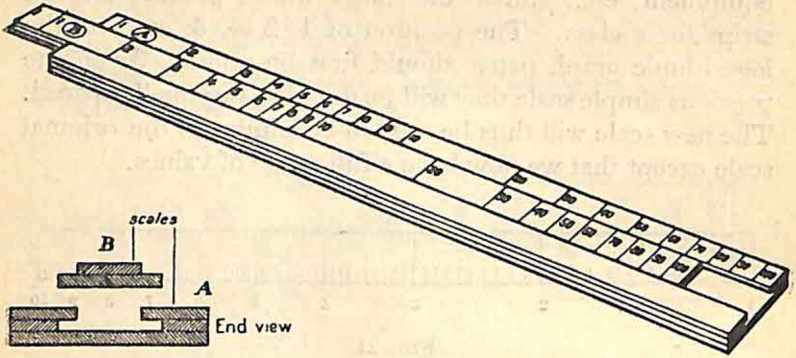

FIG. 23. Slide rule made of card

## DIRECTED NUMBERS

If logarithms are to be used effectively negative characteristics must be taught. Apart from this special need directed numbers are a worthwhile mathematical topic well within the grasp of many pupils. Simple examples of directed numbers may be given before they are formally treated. Comparisons with an average form excellent material. Examples chosen should first be described as above or below the average, and the introduction of + and — signs presents little difficulty. Profiles across a river bed, with land above and below water level, is another useful instance. Levelling (Chapter XII) is an excellent practical introduction.

No mention is made of the term *directed number* during this introductory work, nor is the usage given any particular mathematical significance. The mathematical treatment of directed numbers in VII or VIII standards may very well start with a discussion on the true nature of the numbers which have been used for comparison. It is important that the pupils realize that the signs + and — are given a new meaning. Previously they have indicated an operation to be carried out; now they are used to indicate direction. The term *direction* should not be interpreted in its physical sense only. While the directed numbers used in levelling and profiles certainly show physical directions, the use of a directed number must not be confined to this narrow sense. Profits and losses, deposits and withdrawals, comparisons with an average wage can all use directed numbers. It is often useful to illustrate these graphically, for the visual showing of directed numbers certainly helps many pupils.

Operations with directed numbers are considerably more difficult than the notion itself. Addition and subtraction will be necessary if negative characteristics are to be used in logarithms. Multiplication and division are much more difficult and should be omitted unless there is some specific need for these operations. Addition of directed numbers can be illustrated by finding the total profit of a firm for two consecutive weeks' trading. Here, for example, profits can be written as (+Rs 50), losses as (−Rs 20), and so the situation shows immediately what happens when positive and negative quantities are added. The examples used should be simple, so that the answers are obvious, as follows:

(+Rs 50)    (−Rs 10)    (+Rs 40)    (−Rs 30)
+(+Rs 40)    +(−Rs 25)    +(−Rs 10)    +(+Rs 10)

Adding like quantities, such as $(+3)+(+7)$, or $(-2)+(-8)$, rarely causes much difficulty. When adding positive and negative quantities, the pupils should see that adding $(-3)$ gives the same result as subtracting 3.

The subtraction of directed numbers causes more difficulty than addition. It can be illustrated by comparing profits and losses of two consecutive weeks. The work leads to the rule for subtraction: *Change the sign of the lower line and add.*

## 2. MONEY AND MEASURES

Mathematically the old Indian systems of measures and money were cumbersome and much of our arithmetic teaching had to be directed to them. Calculations using rupees, annas and pies took at least four times as long as corresponding calculations using the metric system. Errors were more frequent owing to the greater difficulty, and to the range of tables to be known. Happily our country has now introduced a decimal system of money in which Re 1=100 paise. This decimal system has decided advantages. Computation has become easy and pleasant.

In the initial stage the pupil may be taught to denote Rs 3 and 43 p. say as Rs 3·43, and read it as three rupees and 43 paise, the 'point' serving *merely* the purpose of separating the whole rupee and the fractional part. The decimal significance will be understood later.

Rs 3 and 20 p. is written Rs 3·20
Rs 3 and 2 p. is written Rs 3·02

remembering the p. is written only as a 'two-figure' number after the point.

*Addition and subtraction.* The addition and subtraction with decimal coinage is very easy. It is just like ordinary addition of integers.

| *Addition* | *Explanation* | | |
|---|---|---|---|
| Rs 3·27 | Rs 3·27 | = | 327 p. |
| Rs 5·02 | Rs 5·02 | = | 502 p. |
| Rs 3·60 | Rs 3·60 | = | 360 p. |
| Rs 11·89 | Total | | 1189 p. |
| | | = | Rs 11·89 |

| *Subtraction* | *Explanation* | | |
|---|---|---|---|
| Rs 12·73 | Rs 12·73 | = | 1273 p. |
| Rs 3·07 | Rs 3·07 | = | 307 p. |
| Rs 9·66 | Balance | | 966 p. |
| | | = | Rs 9·66 |

| *Multiplication* | *Explanation* | | |
|---|---|---|---|
| Rs 73·25 | Rs 73·25 | = | 7325 p. |
| × 7 | ∴ Rs 73·25×7 | = | 7325 p. × 7 |
| | | = | 51275 p. |
| Rs 512·75 | | = | Rs 512·75 |

Notice in the multiplication, the point is ignored and the multiplication is carried out as with integers and then in the result last two digits are marked off as paise leaving the remaining digits to express rupees.

| *Division* | *Explanation* | | |
|---|---|---|---|
| (1) Rs 47·74 ÷ 7 | Rs 47·74 | = | 4774 p. |
| | $\therefore \frac{\text{Rs } 47{\cdot}74}{7}$ | = | $\frac{4774\text{ p.}}{7}$ |
| 7 ) Rs 47·74 | | | |
| Rs 6·82 | | = | 682 p. |
| | | = | Rs 6·82 |
| (2) Rs 52·50 ÷ 8 | Rs 52·50 | = | 5250 p. |
| | $\therefore \frac{\text{Rs } 52{\cdot}50}{8}$ | = | $\frac{5250\text{ p.}}{8}$ |
| 8 ) Rs 52·50 | | = | 656 p. (nearly) |
| Rs 6·56 (nearly) | | = | Rs 6·56 (nearly). |

*Decimalization of money.* Under our scheme of decimal coinage decimalization can be done at sight

| Rs | p. | |
|---|---|---|
| 5 | 70 | = Rs 5·70 |
| 5 | 7 | = Rs 5·07 |

*To find 3 % of Rs 165·90*

3 per cent of Rs 165·90 = $\frac{3}{100}$ of Rs 165·90
= 3 × Re 1·659
= Rs 4·977
= Rs 4·98

## INTEREST†

There is considerable argument as to whether simple interest does, in fact, exist. If the interest is left in an account at the end of each year, it automatically earns interest, so that compound interest is the result. If it is withdrawn each year, then its total value for any period of years would be that found by simple interest methods.

The difference is probably best illustrated by reference to Post Office savings accounts. Rs 100 invested for one year earns Rs 2·50 interest. If the interest is withdrawn each year, the total interest received is Rs 2·50 in 1 year, Rs 5 in 2 years, Rs 7·50 in 3 years, etc.

If the interest is left in the account we can find the interest in the following manner:

*In the 1st year* Money in bank = Rs 100
2½% interest = Rs 2·50

*In the 2nd year* Money in bank = Rs 102·50
2½% interest = Rs 2.56

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† See also page 64.

*In the 3rd year* Money in bank = Rs 105.06
2½% interest = Rs 2.63
*In the 4th year* Money in bank = Rs 107.69 and so on.

The total interest is Rs 2·50 in 1 year, Rs 5·06 in 2 years, Rs 7·69 in 3 years, etc.

The graphs of simple interest and compound interest also show the difference between them. Simple interest is an instance of direct proportion; the interest received varies directly with the number of years the principal is left in the bank. The graph is a straight line. The graph of compound interest is a steepening curve.

The method of evaluating compound interest shown above is tedious when applied to a long period of time. The use of the compound interest formula, $A=P\left(1+\frac{r}{100}\right)^{n}$, demands a better understanding of the use of logarithms than most school children possess. By far the simplest method of dealing with compound interest problems is through the use of the compound interest tables found in some table books. An example of the use of these tables is given below:

*Find the compound interest on Rs 1610 at 3½ per cent per annum for 3 years.*

*From table, Re 1 at 3½ per cent p.a. in 3 years will amount to Re 1.10872*

*∴ Rs 1610 ,, ,, Re 1·10872 × 1610*
*= Rs 1785·0392*

*Interest = Rs 1785·0392 – Rs 1610*
*= Rs 175·0392*
*= Rs 175·04*

The multiplication in the second line of the solution (Re 1·10872 × 1610) must be carried out by normal long multiplication methods; 4-figure logarithms are not sufficiently accurate in this case.

## PRACTICAL MONEY OPERATIONS

Work in the four rules in money will have been started in the primary school and, apart from remedial work, all that is necessary in the secondary school is to extend the range of these processes. A large number of practical extensions will probably be new to the pupils. These may include: hire-purchase payments and instalments; deposits; banking accounts and interest; discount; foreign money; house purchase; budgets; insurance; rates and taxes; wages and time-sheets; balance-sheets.

Operations involving money acquire greater significance when they are shown to be of real practical value, and all the foregoing extensions can be incorporated in the activity described in Chapter XI.

### *Metric Measures and Weights*

In our country, besides following the decimal coinage, we have adopted metric measures and weights. Nearly four-fifths of the world has 'gone metric' and we are now falling in line with them. The advantage of this uniform decimal system for weights, measures etc., is that it makes calculation easy and saves time and labour.

You may for example compare the computation involved in the two following questions.

| | *A* | *B* |
|---|---|---|
| (1) | What is the cost of 3 viss 15 palam of ghee at Rs 9-7-6 per viss? | What is the cost of 5·20 kg of ghee at Rs 9·44 per kg? |

| | A | B |
|---|---|---|
| (2) | What is the cost of $2\frac{5}{8}$ yards of voil at Rs 2-15-3 a yard? | Find the cost of 3·6 metres of cloth at Rs 2·96 a metre. |

## TIME

The nature of time is difficult to understand and difficult to express, but the measures of time are in very common use. It is interesting to notice that these measures are very old and are largely of Babylonian origin. They are also universal. The units of length, weight, etc., vary from country to country, but the same system of time is found everywhere except among a few primitive peoples. Before the invention of the mechanical clock, day and night were often each divided into 12 hours. The day hours in summer were long, in winter they were short. The difficulty of making a mechanical clock which could change the length of its hours brought about the end of this system, sometimes known as ' Italian hours '.

Some post-primary children are still inaccurate in telling the time from a clock, and many will not comprehend times stated in the form 8.50. Besides the normal telling of time from the clock, and work dealing with intervals of time, the 24-hour clock, time-tables, and the calendar should be treated. An increasing use is made of the 24-hour clock and many children will have some acquaintance with it. The time 6 a.m. is written 06.00 or 6.00, and it is quite commonly referred to as ' six hundred hours.' There are obvious dangers in this for children, and ' six hours ' is to be preferred. Calculations in time are often easier when 24-hour clock times are used.

A table of days in the year can give useful exercise to able pupils.

| 1 Jan. | 1 Feb. | 1 Mar. | 1 Apr. | 1 May | 1 June | 1 July | 1 Aug. | 1 Sept. | 1 Oct. | 1 Nov. | 1 Dec. | 1 Jan. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 31 | 59 | 90 | 120 | 151 | 181 | 212 | 243 | 273 | 304 | 334 | 365 |

From the table we can find the number of days between any two dates. A grading of examples will be necessary, e.g.

1. 1 Jan. to 1 Mar. (59)
2. 1 Apr. to 1 Aug. ($212-90=122$).
3. 1 Jan. to 4 July ($181+3=184$).
4. 3 Jan. to 15 Sept. ($243-2+14=255$).
5. 1 Mar. to 17 Aug. ($212-59+16=169$).
6. 19 Jun. to 10 Oct. ($273+9-(151+18)=113$).

## COMPOUND UNITS

Compound units, such as kilometres per hour, provide much greater difficulty than the simple units of measure. Indeed, most adults can use them correctly only in the simplest cases. Averages of speeds are often confused; advertisements can be found in motoring periodicals of devices 'guaranteed to increase petrol consumption.' (It is probable that many readers misread this statement and obtain from it precisely that meaning which the advertiser intended but failed to express.) It is useful to discuss with the class exactly what such statements mean, for a number of compound units are in general use and should be understood. It is as well if the pupils realize that they cannot be handled in precisely the same way as simple units. Calculations involving compound units need to be carefully considered.

On the whole the teacher is wise to assume that students at the level of say, standard VII have little knowledge of compound units and no skill in handling them. It is one thing to be able to speak of miles per hour; it is quite another matter to understand the true nature of speed.

It is probable that the meaning of compound units cannot be satisfactorily explained to some pupils, but that an inward grasp of their significance can be given by illustrating them in a variety of simple cases. As with all fundamental ideas the treatment should take place over a long period of time.

Besides the more familiar *miles per hour*, *miles per gallon*, *kilometres per litre*, wage rates (*rupees per hour*), etc., some of the less usual forms might be treated. For example, the term *man-hours* of work is increasingly used. The pupils should find little difficulty in realizing that the amount of work done depends both on the number of men engaged upon the work, and on the number of hours they work. Forty *man-hours* of work are done by one man working for 40 hours, or by 2 men working for 20 hours each, or by 4 men working for 10 hours each, etc.

## SQUARE ROOT

The subject of square roots is usually introduced by a problem thus: The area of a square is 9 sq. cm. What is the length of the side? With a knowledge of the multiplication table the answer is at once given as 3 cm.

But suppose the area of the square is 10 sq. cm. In this case the length of the side is not so easy to say.

The answer is between 3 cm and 4 cm. Perhaps 3 cm and a little extra bit say x cm (x being less than 1).

You will notice (Fig. 24) that the unshaded portion has an area 1 sq. cm (10–9). The unshaded portion can be cut and arranged as a single rectangular strip length $(6+x)$ cm and breadth $x$ cm.

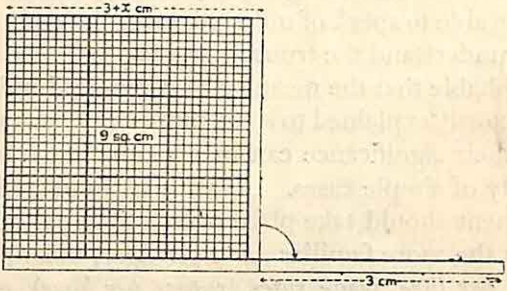

Fig. 24

$$\therefore x(6+x) = 1$$

from which we get $x = \dfrac{1}{6+x}$.

By trial, $x$ will be found to be over 0·1.

Generalizing, if the square root of a number $N = a +$ an extra bit $b$, $b = \dfrac{(N-a^2)}{2a+b}$,

which can also be derived from the formula

$$N = (a+b)^2 = a^2 + b^2 + 2ab = a^2 + b(b+2a).$$

This leads to the theory of actual extractions of the square root of a number.

## VI

# GRAPHS AND DIAGRAMS†

GRAPHS and diagrams have been used increasingly to present information during the last few decades, but the very great value of graphical work in the mathematical education of the ordinary child is not yet fully realized. The graphical method gives a more vivid presentation of information than any other method. It is a presentation with particular advantages for average and below-average children. It can also provide an intermediate stage between a set of practical situations and the abstract mathematical idea contained in these situations. In this latter respect its importance cannot be over-emphasized.

The point merits an example. The child's first understanding of vulgar fractions derives from a large number of practical examples of fractional parts. Diagrams can form

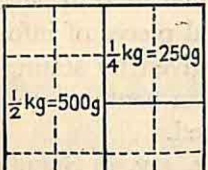

FIG. 1

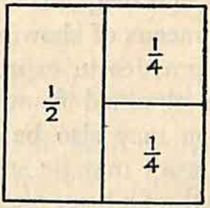

FIG. 2

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† The term GRAPH is commonly used to describe many diagrammatic forms of representation which the mathematician would not readily accept as graphs. It is not easy to define a *graph* in exact terms, but it may generally be taken that a true graph shows a continuous relationship between two or more variable quantities. Other visual forms, such as isotypes or sectograms (see under *Types of graphs*), have their particular values but they cannot be compared with the true graph as a fundamental mathematical tool.

an intermediate stage between this concrete usage and the power to deal with the abstract quantities $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{3}$, etc., which have no separate real existence, but are ideas arising from experience. The first diagrams used should illustrate fractional parts of a unit of measure. Fig. 1 shows the fractional parts of 1 kg. At this particular stage the child is learning to show a measure and its parts in this new diagram form. When this usage has been mastered the diagram can show the fractional parts only without reference to any measure (Fig. 2). The diagram thus becomes a generalization in visual form of the child's previous experiences and takes him one step further towards accepting the abstract idea.

## GENERAL TREATMENT OF GRAPHS

The following aspects of graph work might receive treatment in the higher classes:

### (*a*) *Reading information from graphs*

Graphs and diagrams are commonly used in newspapers, etc., as a means of showing essential pieces of information. Valuable practice in expression is given by stating the information obtained from a graph in sentences; tables of information may also be constructed.

Graph work may be started early, say, in Standard VII. The growth of a bean plant is an excellent topic with which to arouse initial interest, and if the bean is tended with reasonable care the growth of the plant is quite satisfactorily regular. A method of introducing the record of growth, which even the least able can follow, is to place a large sheet of squared paper on a wall with the plant on a shelf in front of it. The plant is moved along a fixed distance daily, and its height recorded by drawings on the squared paper.

In this way the record in its final form shows a diagram of growth at a series of fixed points in time. It is unlikely that readings will be taken on Sundays and spaces should be left for these days. The pupils may be asked to consider the other readings and to estimate the height of the plant on these days. The teacher should realize that, when estimating the heights on Sunday, we are assuming a regularity of growth which may not exist in fact. The graph gives a first approximation to the height on these days, an approximation which cannot be improved upon in the absence of actual readings. It is probably wiser not to attempt to discuss this point with the class at this early stage.

After their first introduction the development of graphs may be carried out almost entirely by usage. Ordinary class work may be extended with advantage by making a collection of graphs from newspapers and other sources. These may be used in a series of individual work-cards, each containing a graph, together with questions on it. The series can be supplemented with cards containing tables of information and from these the children may be asked to construct their own graphs.

### (*b*) *Displaying information by graphs and diagrams*

This is probably the most common form of treatment in many schools. This work should, however, go beyond mere exercises in the drawing of graphs. It often happens that relationships are seen more readily when information is presented in visual form and the work acquires a deeper significance. What the more intelligent child may see from the data alone is shown to the ordinary child by the visual presentation. The following steps might be taken when displaying information by a graph. At each step the children will need guidance for some time.

(*i*) A discussion might take place as to the most suitable form of graph. Where there is a difference of opinion it is worth while allowing groups within a class to construct different graphs from the same information. In this way the particular advantages of individual types of graphs become clear.

(*ii*) Having decided the form of graph, convenient scales must be fixed to show the range of quantities involved. Long practice is needed before children are capable of doing this for themselves. Nevertheless, some discussion should take place on this point, for it is only through repeated discussion and practice that children will learn the principles involved in the fixing of scales.†

(*iii*) The graph is drawn.

(*iv*) Some use is made of the graph, e.g. further information is obtained from the graph, or a statement is written giving some information which the graph shows clearly.

The mathematics period is by no means the only possible occasion for such work. Geography, social studies, and science are other lessons in which the need to construct and use graphs may arise. As far as possible the information and observations used should be collected by the pupils. This may be data recorded at first hand, such as a traffic census or a child's own weather observations; it may be the result of reading research, population statistics, or rail or bus time-table work.

At a later stage in the treatment of work in this section, pupils may be asked what kind of graph curve to expect from certain conditions and situations. The quick drawing of rough graphs, such as that showing the age and height of a person, is a valuable exercise occupying very little time.

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† Further information on types of graphs, scales, etc., is given on pages 92-96.

With practice a pupil may well find himself able to express a set of conditions more easily in graphical form than in words. At least we shall have given him a new mode of expression.

### (*c*) *Graphs as an aid to calculation*

This is another important usage for the ordinary child. Quite often a graph can relieve much tedious calculation or, in some cases, provide a method of solution to a problem which is beyond the ability of the child to solve in any other way. For example, it is only the extensive use of graphical methods which puts the work on *Sun and shadows* (page 208) within his grasp. The graphs in this section can be divided into two types, but with no sharp division between the two.

#### (*i*) *Graphs showing non-mathematical relationships*

In the first type there is no definite mathematical relationship between the quantities involved, e.g., growth of a plant; age and height of a child; times and distances of a bus or train journey. A series of observed or known points are plotted and the graph drawn. By interpolation (taking readings from the graph between the plotted points) further information can be found. In doing this we are assuming a regularity which may not exist. Nevertheless, it is impossible to obtain a more satisfactory answer by calculation, and in many cases the calculation would be too difficult for pupils.

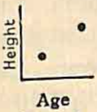


FIG. 3

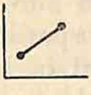


FIG. 4

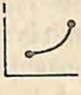


FIG. 5

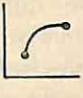


FIG. 6

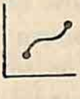


FIG. 7

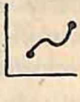


FIG. 8

At some time the children should be made aware of the assumptions taken in this kind of work, although the ideas involved would cause difficulty if introduced too soon. Standards VII or VIII are probably early enough to discuss these points, and rough graphs could profitably be tackled at the same time. For example, two points might be plotted on the graph of the age and height of a child (Fig. 3). The line joining the points could be any one of a wide range (Figs. 4-7), although some forms are improbable (Fig. 8).

### (*ii*) *Graphs showing mathematical relationships in practical situations*

In a large number of practical situations some definite mathematical relationship is involved, e.g. changing miles to kilometres (a case of direct proportion); the area of a square of given side, etc. The child should be given a long experience of these before any more formal graphs, such as $y=3x+2$, are tackled. Indeed, for most school children, the mathematical graph in a practical situation is as far as they can profitably go.

Children will not at first distinguish these graphs and the relationships they show from those of type (*i*). It is a mistake to assume that children realize that a direct proportion relationship will give rise to a straight-line graph, and that in order to draw the graph all that is needed is to join two plotted points by a line drawn with the aid of a ruler. Direct proportion is but one of many forms of relationships and it is most likely that, at the time when we can first introduce the straight-line graph to the child, he is not clearly aware of the different classes into which these relationships can fall. For him it is quite possible that direct proportion is not precisely established in its own class distinct from a case such as the weight and height of a person.

A careful approach is needed to ensure that the children do realize this for themselves and so obtain a full and true insight into the mathematical idea implied. In the first cases a whole series of points should be plotted, sufficient to show clearly that a straight line is obtained. This practice should be continued until the class obviously expect that a straight line will result. Then is the time to try to discover why such a relationship gives a straight line, and to discuss methods of drawing the graph more rapidly with fewer points.

Certain factors should be watched when drawing graphs from experimental results, etc., where a definite relationship is apparent. In some cases the majority of points plotted are seen to give the smooth curve which might be expected, but one or two points lie outside this curve. When possible these particular values should be rechecked, as it is almost certain that error has occurred. If it is not possible to recheck, the points should be disregarded, and the graph line drawn through the other points.

Sometimes the points will be seen to lie in approximately a straight line. In this case a straight line should be drawn to pass between the points as nearly as possible. This gives a much truer picture of the relationship than does a series of lines joining each point to the next.

### (*d*) *Formal mathematical graphs*

Only the more able pupils can receive any benefit from a study of formal mathematical graphs, and these should be confined to a few simple types, preceded by at least one year's work on graphs of type *c* (*ii*). The children should then be able to recognize cases of direct proportion and realize that they all give straight-line graphs. Drawing the graphs of linear algebraic equations, e.g. $y=2x$, $y=5x+3$, etc., will help to establish the relationship in a more formal setting. The pupil should be able to associate direct proportion with the linear equation, and know the form of graph obtained.

Cases of inverse proportion are common in practice, and drawing the graph of $y = 1/x$ will help to distinguish this family.

The set of parabolas given by quadratic equations, $y = x^2$, $y = 2x^2 - 1$, etc., form another distinct group and again practical examples are not difficult to find. In this case, however, the plotting of the algebraic graphs involves the squaring of negative quantities. One possible solution to this difficulty is through the plotting of bridge shapes, which can often be expressed by equations of a simple nature. For example, the chains of a suspension bridge at Bristol in England closely follow the curve $y = x^2/1600$, taking the centre of the bridge as origin. The length of the bridge is 201.168 metres, so the values of $x$ ranges from $-100.584$ to $+100.584$. If positive values of $x$ are plotted, one half of the bridge is given. The other half can then be drawn by symmetry. Curves of this type always arouse interest, and the method is a quite satisfactory way of obtaining the graph of $y = x^2/1600$, if the limitations of the pupils are taken into consideration.

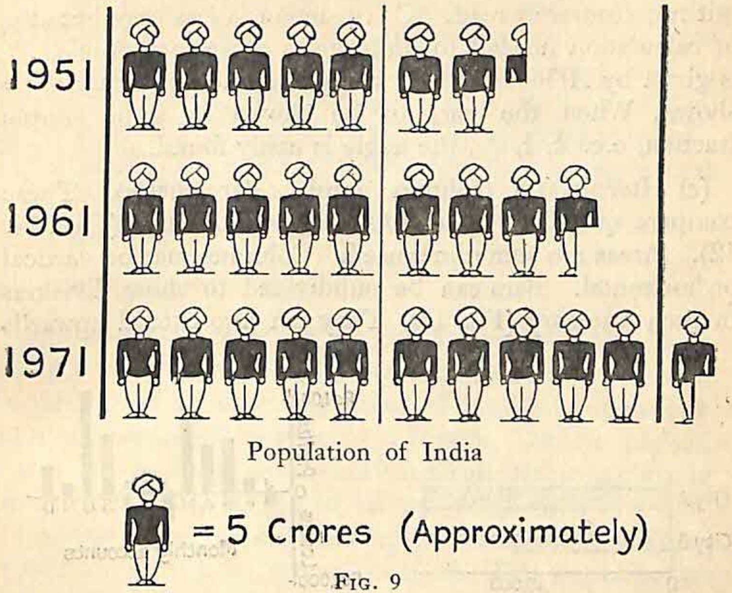
More approximate equations of bridge shapes are given by W. W. Sawyer in *Mathematicians Delight* (Penguin), page 119.

## TYPES OF GRAPHS

This section shows various types of graphs and diagrams and gives brief descriptive notes.

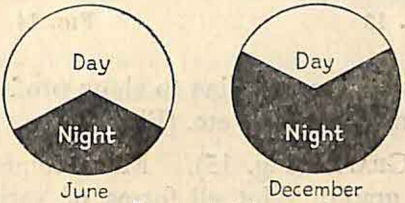
(*a*) ISOTYPE (Fig. 9). A common and easily read form of presentation. Vivid and popular with children. Not as useful or exact for persons with experience in reading graphs as some other forms.

(*b*) SECTOGRAM ($\pi$-chart) (Fig. 10). Especially useful for showing parts of a whole by using sectors of a circle. Easily



Population of India

FIG. 9



June December

Day and night

FIG. 10

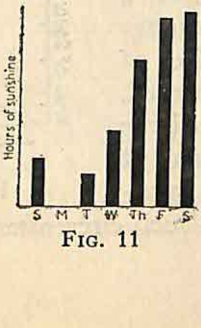


FIG. 11

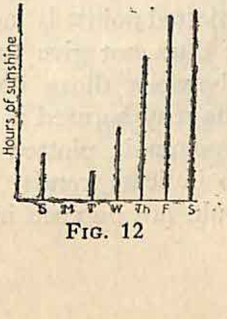


FIG. 12

but not accurately read. Construction is less easy because of calculation needed to find angles of sectors. Angle, $A$, is given by $A/360° = P/100$, where $P$ is the percentage to be shown. When the part to be shown is some simple fraction, e.g. $\frac{1}{4}$, $\frac{2}{3}$, $\frac{7}{12}$, the angle is easily found.

(*c*) HISTOGRAMS (column graphs, bar charts). These compare quantities by lengths of lines or columns (Figs. 11, 12). Areas are sometimes used. Columns may be vertical or horizontal. Bars can be subdivided to show divisions in each category (Fig. 13). They can also extend upwards or downwards from a zero line to show profit and loss, or deviations from an average, etc. (Fig. 14).

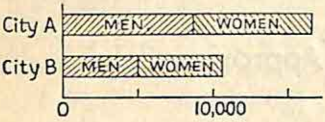


FIG. 13

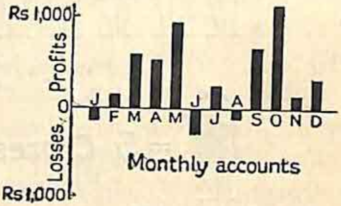


FIG. 14

(*d*) LINE GRAPH (Fig. 15). Most common form of mathematical graph. Not all forms of variation should be shown by continuous line graphs. In a record of daily attendances, a line joining plotted points is meaningless, since it does not give values of attendance between those observed. A joining line may be used if a large number of points is plotted and it is wished to indicate trends. Interpolation should not be used in such circumstances.

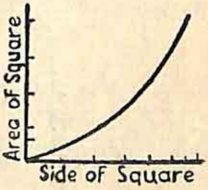


FIG. 15

If the two quantities to be plotted are considered, it will be found in many cases that one is dependent upon the other; for example, the weight of a child depends (partly) upon its age, the elevation of the sun depends on the time of day, the length of a coiled spring depends on the load placed on it. The dependent quantity should be plotted on the vertical axis, the independent quantity on the horizontal axis.

## GRAPH SCALES

The scale to be used needs careful thought. The range of values to be plotted should be noted and compared with the number of units on the scale. From this comparison the most convenient near scale is chosen. It is important to use a convenient scale to avoid much calculating in plotting and reading. If a length of 10 cm, subdivided into 100, is available to represent a maximum value of 85, the first comparison gives 1 cm representing 8·5. This would not be convenient, and a scale of 1 cm = 10 units is to be preferred.

The scale need not extend from zero to the maximum to be shown, although careful reading is then needed to prevent being misled (Figs. 16 and 17). The fact that the full

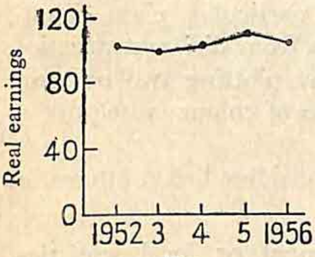


FIG. 16

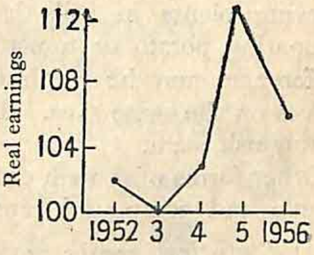


FIG. 17

range is not shown can be indicated by a break in the scale or across the graph (Figs. 18 and 19).

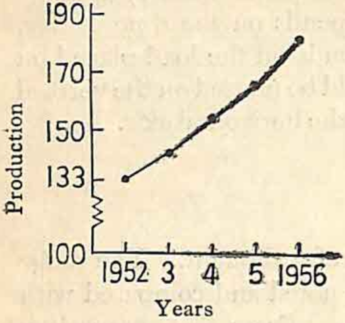


FIG. 18

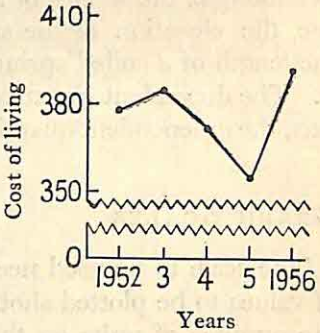


FIG. 19

## SUMMARY OF USES OF GRAPHS

Here is a list of instances where graphical methods are profitable. It is by no means exhaustive.

(*a*) *Diagrams used as visual aids*, e.g., decimal notation (page 38, Fig. 2); decimal fractions (page 56, Figs. 10, 11); vulgar fractions (page 85, Figs. 1, 2).

(*b*) *Growth.* In the third and fourth years, graphs of growth under varying conditions are of interest; for example, growing plants in soils lacking particular plant foods; comparing potato or tomato crops from different varieties. Differences may be emphasized by plotting two or more curves on the same axes. The use of colour is helpful to distinguish them.

Other forms of growth can be considered, e.g. interest at simple and compound rates.

(*c*) *Statistical graphs* from national or local statistics. Also from pupils' own observations, a traffic census, weekly marks, etc.

(*d*) *Graphs from astronomy, geography, science, etc.*, e.g. shadow lengths (page 208); sections and profiles; tide heights and dates.

The number of hours of daylight can be plotted at fortnightly intervals throughout the year; the information may be obtained from a diary. The same information can be shown by plotting the times of sunrise and sunset against times of year. This gives two curves between which lie the hours of daylight. Average temperatures may be plotted on the same time axis as the hours of daylight. This requires the use of two different vertical scales which are better separated from each other, one on the left and the other on the right of the paper. Colour is also helpful.

Graphs can also show experimental results, e.g. length of a spring and its load; the cooling curve of a heated liquid, etc.

(*e*) *Ready-reckoner graphs*, e.g. changing miles to kilometres, gallons to litres, viss to kilograms, shillings to rupees; areas of square from side.

(*f*) *Graphs from bus and rail time-tables.*

(*g*) *Rough graphs.* Before a graph is drawn it is often desirable to discuss its probable shape and to attempt a rough drawing. A comparison is later made with the plotted graph.

(*h*) *Graphs of bridge structures* (see page 92).

(*i*) *Sun and shadows* (see page 208).

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# VII

# GEOMETRY

The geometry of Euclid is largely an exercise in deductive logic and many of the geometrical properties were first discovered practically by the Egyptians. The Greeks provided formal proofs of these properties and Euclid gathered the proofs together in a logical whole, using geometry as a medium to show the power of logic. From a small number of first assumptions an amazing range of deduced properties is shown. For the High School course we should go back to the Egyptians, thinking of geometry as a conscious discovery of the space in which we live and the relationships we find in it. That this form of geometry has its own high standards is shown by the achievements of the Egyptians. The pyramids, for example, were built to an amazing degree of precision. Measurement shows that the angles of the Great Pyramid, built about 2,600 B.C., differ from 90° by not more than $\frac{1}{18}^{\circ}$; the sides deviate from north, south, etc., directions by not more than $\frac{1}{12}^{\circ}$; the sides of over 228 metres in length differ by less than 20 cm, an error of less than 0.1 per cent. These standards of accuracy are found on a monument whose base covers over 5 hectares, and which was built with primitive tools and instruments.

## THE CHILD'S GEOMETRICAL KNOWLEDGE

From the time when the baby, a few months old, becomes aware of its surroundings, he is perceiving and absorbing spatial relationships and properties. The store of primitive geometrical ideas acquired from daily experience and observation is surprisingly large; we must help the child to become more fully aware of these notions, and to extend their range and use. He will have a notion of distance and will have observed shapes, both plane and solid. A notion of direction will be established; horizontal and vertical direction will be well known, together with something of the relationship between them. It is most unlikely that he will use the technical terms to describe them, or be able to express himself in exact language, but the fundamental ideas are there. Part of the teacher's task is to encourage the pupil to use the language of geometry to express these commonly observed relationships and properties. He will possess a sense of symmetry, a valuable starting-point for many geometrical properties. Lines and planes will be known although their geometrical abstractions will still be foreign to him. He will be well acquainted with parallel lines. Apart from his observation of the world around him with its constant use of geometrical form and pattern, he will also have used geometry, consciously and unconsciously, in the drawing, and building, and making, of which all children are fond.

## THE STUDY OF GEOMETRY

If we are to use effectively the child's store of subconscious knowledge, our treatment of the subject must be specially directed towards this end. A treatment which consists largely of formal mensuration and drawing of geometrical shapes will fail, since the gap between it and the child's perception of space is too wide to be easily bridged. There is a need in the secondary school geometry curriculum for a medium, or carrier, which will display the spatial properties we are trying to teach and, at the same time, link them effectively with the child's world. The medium may be one such as surveying, which can be taken throughout the years of the school course; or it may occupy one or two lessons only, for example, the making of a model. Whatever carrier is used, it is important that it is not used for its own sake only; the pupils must be led to discover the geometry implicit in its use. When children discover geometrical truths for themselves, geometry becomes one of the most exciting and fascinating of all school subjects. Much thought should be given to providing appropriate situations.

Many able pupils will be found to have intuitive knowledge of certain geometrical ideas. When making or using a plan or map the child is accepting the principles of similarity. The shadow stick method of finding heights (page 184) rarely needs explanation. The pupils instinctively recognize the principles involved when applied to simple cases which use simple numbers. The equality of vertically opposite angles is a property which few children fail to observe. This property is shown more effectively by a pair of rods on a common centre pivot. The pupils are told to watch a pair of opposite angles as one of the rods is moved around the pivot. In this way the whole range of vertically opposite angles is seen. The property then becomes a true generalization instead of a statement about one particular pair of opposite angles. This method can obviously be applied in other cases.

In such examples of readily observed properties, a practical situation can well be the beginning of a more generalized treatment of the idea. Other notions, such as Pythagoras' theorem, or the value of $\pi$, are not intuitive, and the teacher must lead the child to discover these. In all work the value of observation must be stressed; geometrical properties can be observed and it is through observation that the geometrical truths are discovered. The drawing of circle patterns can lead the observant child to the idea that the circumference of a circle is a little more than three times its diameter. A more exact ratio might then be found in a variety of ways.

The study of geometry should arise, therefore, from some practical situation or use: surveying, architecture, navigation, practical drawing, model making and construction of all kinds, making and using maps and plans, spacing and arrangement, the shadow stick, the marking out of a games pitch, etc. A study of some familiar shapes: bricks, wooden blocks and cubes, balls, tins of all kinds, cones, etc., can provide a wealth of material. The study should be of the actual objects themselves, not pictures of them. The work should include measuring, description, and drawing. In suitable cases, e.g. cardboard boxes, a box may be unfolded to show its net. The nets needed to provide variations of size and shape can then be discussed and made.

It is from such contexts that the child begins to generalize particular shapes and identify them with their correct names. He will distinguish a rectangle and name it correctly. Discussion will bring out some of its properties: the opposite sides are equal, the diagonals are equal and bisect each other, each ' corner ' is a right angle. If this is done before angle measure is introduced, the term '*true square corner*' is useful. A paper set-square can be used for testing. [The construction of a paper set-square is shown in Figs. 1 and 2. A piece of paper is folded (Fig. 1), and then folded again so

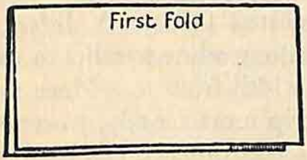


FIG. 1

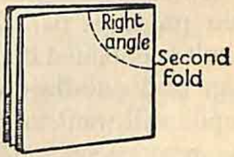


FIG. 2

that the first fold lies along itself (Fig. 2).] The drawing of a rectangle can follow, using some of its properties for drawing, and applying others as tests of accuracy.

High school geometry should not consist only of a study of the properties of common shapes. There are other important geometrical ideas, such as loci and envelopes, which are well within the grasp of many high school pupils. Their treatment can often be incidental to other work; the bisector of an angle, and the perpendicular bisector of a straight line are both examples of loci.

## GEOMETRICAL DIVERSIONS

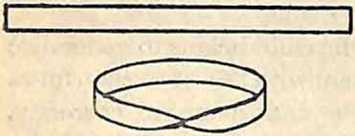


FIG. 3

Less utilitarian, but nevertheless enjoyable, are diversions such as pattern making, tangrams (page 291), curve-stitching (page 256), etc. The Möbius strip rarely fails to fascinate. A strip of paper is twisted once and the ends fastened together to make a loop (Fig. 3). This band has two sides but only one surface. This can be shown by starting from a marked point and drawing a pencil line around the band. A continuous line traverses both sides of the band before arriving back at the point. The edge of the band is also one continuous line. The result of cutting the strip all round along a centre line always surprises. Cutting again along a centre line produces two linked and twisted loops. A different result is obtained if a band is cut along a line parallel to one edge and one-third of the band width from it. Most able pupils will want to look at the strip more closely, to repeat the experiments, and see why these curious things happen. It serves as an excellent stimulus to spatial perception.

Patterns can be made by drawing, using squared paper, or ruler and other instruments. A greater element of enjoyment is attained through colouring the patterns, especially for the less able children. Rectangles or squares of coloured gummed paper can be folded in different ways to give different shapes. These can be cut out and stuck on a sheet of paper to form patterns. The folding can lead to a discussion on symmetry and so to a more effective use of folding.

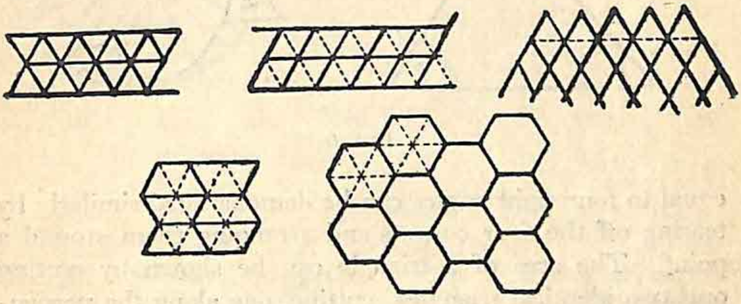


FIG. 4

Some work on patterns can lead to a study of the 'space filling' shapes. The practical problem of tile shapes to cover a floor will show that, besides the square or rectangular tile, we can use equilateral triangles. By combining the triangles in various ways we can fill the space with tiles the shape of a rhombus, trapezium, hexagon, etc. (Fig. 4). On the other hand, if we use octagonal tiles we find that 'gaps' are left which have to be filled with tiles of another shape (Fig. 5).

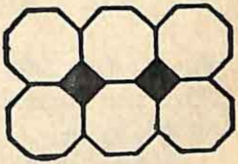


FIG. 5

Following the recognition and definition of geometrical shapes, some demonstrations of further properties can be carried out employing a variety of teaching techniques. Cut-out paper shapes can be most effective. The sum of the angles of a triangle is demonstrated by cutting out a triangle, tearing off the corners, and arranging them as shown in Fig. 6. That the sum of the angles of a quadrilateral is

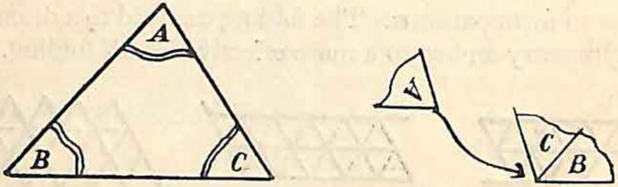


Fig. 6

equal to four right angles can be demonstrated similarly by tearing off the four corners and arranging them around a point. The area of a triangle can be shown by cutting out two identical triangles, cutting one along the perpendicular from apex to base, and arranging to form a rectangle as in Fig. 7. Pythagoras' theorem is illustrated by cutting out graph-paper squares and mounting them on each side of a right-angled triangle (Fig. 8). The area of a circle may

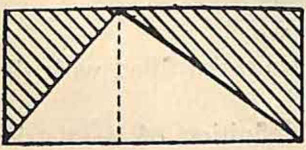


Fig. 7

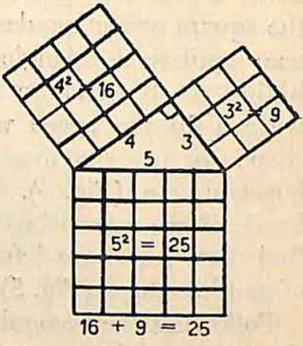


Fig. 8

be found by dividing the circle into a large number of equal sectors, and arranging them to form an approximate rectangle

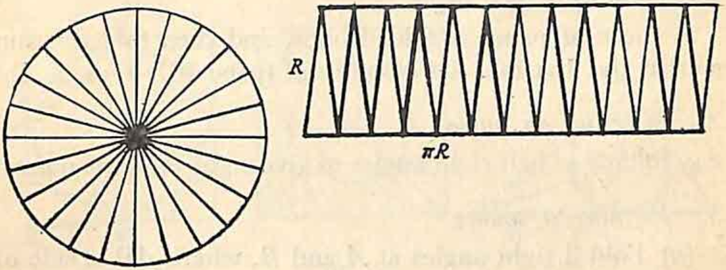


FIG. 9

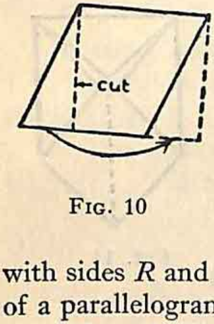


FIG. 10

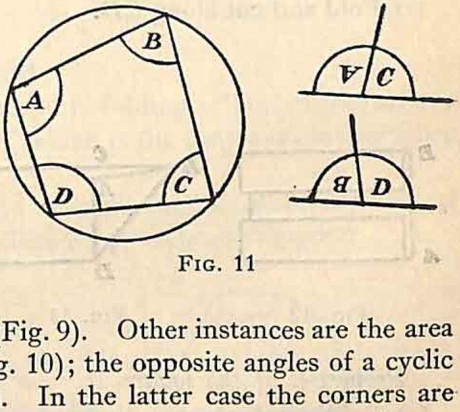


FIG. 11

with sides $R$ and $\pi R$ (Fig. 9). Other instances are the area of a parallelogram (Fig. 10); the opposite angles of a cyclic quadrilateral (Fig. 11). In the latter case the corners are torn off and placed in pairs, with opposite angles together.

## PAPER FOLDING

Drawing constructions to show geometrical properties are well known and commonly used. Paper-folding methods are less common but can show a wealth of geometrical properties. Some of its possibilities are given below.

1. *Forming a right angle*

A piece of paper is folded once, and then folded again so that the first fold lies along itself (page 101, Figs. 1, 2).

2. *Bisecting an angle*

e.g. folding to half right angles to give eight compass points.

3. *Forming a square*

(*a*) Fold 2 right angles at *A* and *B*, where *AB* is side of required square. Cut off surplus strips (Fig. 12).

(*b*) Fold so that side *AB* lies along upper edge of strip (Fig. 13).

(*c*) Fold and cut along *CD*.

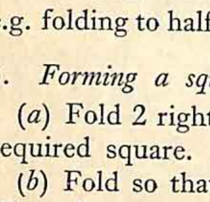


Fig. 12

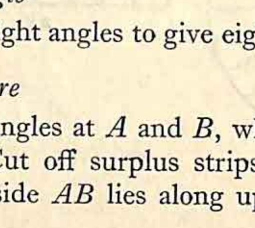


Fig. 13

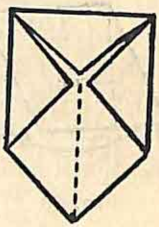


Fig. 14

4. *Properties of the square*

(*a*) Folding along the diagonals shows that they bisect the angles.

(*b*) Intersection of diagonals is centre of square. Folding corners to centre gives an inscribed square, half the area of the original square (Fig. 14). Repeated folding gives squares $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc., of the original square. (Note the series: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$)

(*c*) A regular octagon is obtained by folding to bisect the angles between the sides of a square, *ABCD*, and its inscribed square, *EFGH* (Fig. 15) (e.g. the fold *HJ* bisects ∠*AHE*).

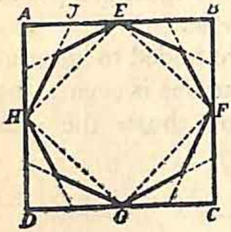


FIG. 15

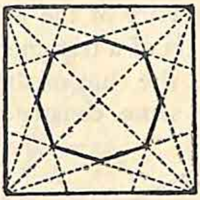


FIG. 16

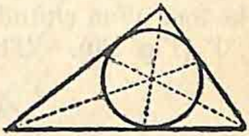


FIG. 17

(*d*) Another regular octagon is formed by dividing each angle of the square into four equal parts (bisecting twice) (Fig. 16).

5. *Properties of triangles*

(*a*) Bisect each angle by folding. The three creases formed meet at a point, which is the centre of the inscribed circle (Fig. 17).

(*b*) The altitude of a triangle can be found by folding (Fig. 18). The three altitudes pass through a point.

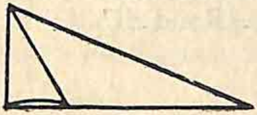


FIG. 18

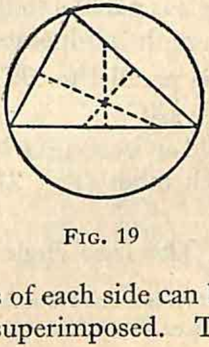


FIG. 19

(*c*) The perpendicular bisectors of each side can be found by folding so that two corners are superimposed. The three perpendicular bisectors of the sides pass through a point which is the centre of the circumscribed circle (Fig. 19).

(*d*) The midpoints of the three sides of a triangle are given by construction 5(*c*). The three medians[†] meet in a point. This is the centre of gravity of the triangle.

(*e*) If the three corners of a triangle are folded to meet at the foot of an altitude, the sum of the angles is seen to be 180° (Fig. 20). The same construction shows the area of a triangle. The whole triangle forms two rectangles, each of area ($\frac{1}{2}$ base) × ($\frac{1}{2}$ height),

$$\text{i.e. Area of triangle} = 2 \times (\tfrac{1}{2}\text{ base}) \times (\tfrac{1}{2}\text{ height})$$
$$= \tfrac{1}{2}(\text{base} \times \text{height}).$$

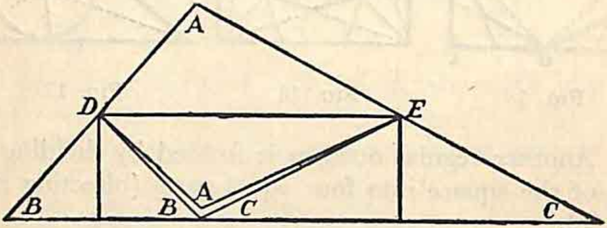


FIG. 20

It also shows that the line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it. *D* and *E* are the midpoints of *AB* and *AC*.

*DE* is parallel to *BC*.
$DE = \frac{1}{2}BC$.

(*f*) Fold an isosceles triangle so that the two equal sides lie along each other (Fig. 21). The following properties can be seen:

(*i*) The base angles are equal.

---

[†] A median is a line from the midpoint of one side of a triangle to the opposite vertex.

(*ii*) The bisector of the vertical angle bisects the base at right angles.

(*iii*) The perpendicular bisector of the base bisects the vertical angle (the converse of (*ii*) ).

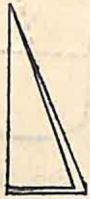


FIG. 21

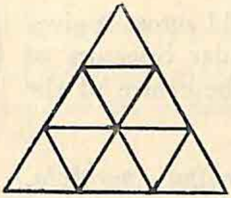


FIG. 22

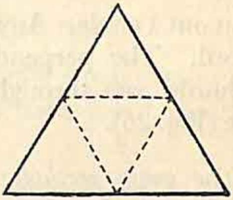


FIG. 23

(*g*) In the case of an equilateral triangle, the altitudes, medians, bisectors of angles, and perpendicular bisectors of sides all pass through the same point.

If the corners of an equilateral triangle are folded to the centre, a regular hexagon is formed (from six equilateral triangles) (Fig. 22). The lines joining the midpoints of the sides of an equilateral triangle form an equilateral triangle (Fig. 23), $\frac{1}{4}$ area of the original triangle. (Four triangles can be folded upon each other.)

### 6. *Symmetry*

If we fold a piece of paper, then cut a shape from one point on the fold to another point on the fold, the resultant shape has one axis of symmetry (Fig. 24).

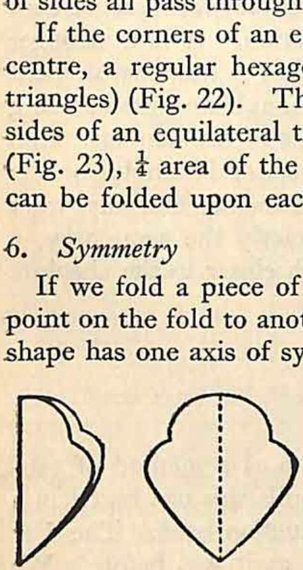


FIG. 24

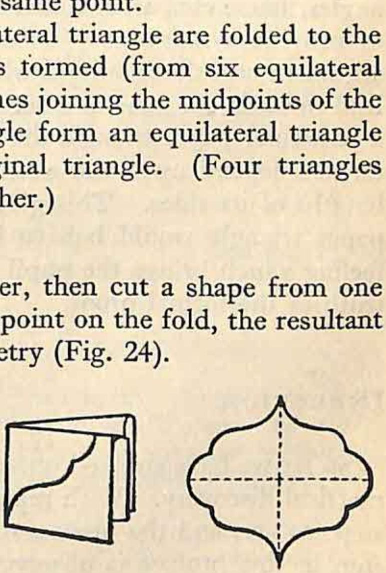


FIG. 25

If two perpendicular folds are formed and the paper cut as shown in Fig. 25, the resulting shape has two axes of symmetry.

7. *Circle*

Cut out a circle. Any fold across it gives a chord. The perpendicular bisectors of all chords pass through the centre of the circle (Fig. 26).

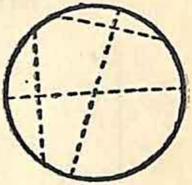


FIG. 26

8. *The conic sections: ellipse, parabola, hyperbola* (see page 249).

For discovering or displaying geometrical properties, methods such as paper-folding are generally to be preferred to measurement. It is not easy for children to measure angles, lines, etc., with sufficient accuracy for measurement to provide a satisfactory demonstration. It is a stronger argument still that methods which do not use measurement lead towards the general case. Although we may be using a particular paper triangle the results of the demonstration do not depend upon the exact values of its angles or the lengths of its sides. This gives the feeling that any other paper triangle would behave in exactly the same way, a feeling which brings the pupil much closer to the absolute truth of the formal proof.

## INDUCTION

So far we have stressed observation as a method of geometrical discovery. With many pupils this can be taken a step farther, and the process of induction used. The first step in this process is observation, as it was before. We must recognize, however, that observation can sometimes lead to false conclusions, and that conclusions should therefore be tested. The method of induction includes:

(*a*) Observation.
(*b*) A generalization or conclusion reached through observation.
(*c*) Testing the generalization in other cases.

The teaching of the sum of the angles of a polygon is given below as an example of the method of induction. (Suitable for standard VIII).

It is assumed that the class know the sum of the angles of a triangle, and the lesson is started by asking them to write down this sum in right angles.

The pupils then draw a quadrilateral, and find the sum of its angles by dividing it with a diagonal into two triangles. (All steps such as this should be elicited from the class, with guidance if necessary, e.g. 'I know the sum of the angles of a triangle. Is there anything I can do to the quadrilateral, which will let me use what I know about a triangle?')

The process is repeated to find the sums of angles of polygons with 5, 6, 7, sides. From the results a table is made:

| *Name* | *A*<br>*No. of sides* | *B*<br>*No. of triangles* | *C*<br>*Sum of angles* |
|---|---|---|---|
| Triangle . | 3 | 1 | 2 right angles |
| Quadrilateral . | 4 | 2 | 4 ,, |
| Pentagon . | 5 | 3 | 6 ,, |
| Hexagon . | 6 | 4 | 8 ,, |
| Heptagon . | 7 | 5 | 10 ,, |

The links between the numbers in the columns are not difficult to see. Inspecting columns A and B shows that each number in column B is two less than the corresponding number in column A. Each number in column C is twice the corresponding number in column B.

This is tested by finding the sum of the angles of an octagon. The number in column A will be 8; in column B will be $8 - 2 = 6$; and in column C, $6 \times 2 = 12$, i.e. the sum of the angles of an octagon is 12 right angles. The result is checked by drawing an octagon and dividing it into triangles.

Another similar test follows, and then a generalized statement is obtained. If there are $n$ sides (column A), there will be $(n-2)$ triangles (column B); the sum of the angles will be $2(n-2)$ right angles (column C).

The teacher should note that this is not a proof; it is a discovery of the property by induction.

## DEDUCTION

Very simple examples of deduction might be used in suitable cases. For example, when dealing with parallel lines, the equality of alternate angles is not so readily observed as that of corresponding angles. It can, however, be deduced quite simply. In Fig. 27:

$x = 70^\circ$ (corresponding angles)
$y = 70^\circ$ (vertically opposite angles)
$\therefore\ x = y$.

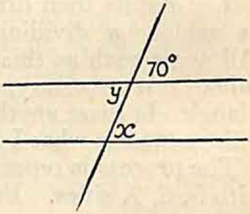


FIG. 27

## THE CONTENT OF THE GEOMETRY COURSE

It is not suggested that all the following work should be done, but that a selection might be made. The list is of basic geometrical properties and notions, and the teacher must find appropriate situations and uses for them. A few examples of such situations are given at the end of the list. The order is not a teaching order.

*Direction and angle measure.* Horizontal and vertical lines and planes. The right angle; acute, obtuse, and reflex angles. Compass points. Measuring in degrees; use of protractor; bearings. Angles at a point, vertically opposite, supplementary, etc. Parallel lines and their properties.

*Triangles.* Side and angle properties of scalene, isosceles, and equilateral triangles. Sum of angles; the exterior angle. Area. Construction of triangles, given three sides, two sides and included angle, one side and two angles. Bisectors of angles and sides, altitudes, medians; the inscribed and circumscribed circles. Pythagoras' theorem and its uses; construction of right angles (e.g. through 3, 4, 5 triangle); finding square roots; solving right-angled triangles.

*Areas and properties of other rectilineal figures:* rectangle, parallelogram, quadrilateral, regular and irregular polygons.

*The circle.* Circumference and area of a circle; area of sector. Angle properties: angle in a semicircle, angles in the same segment, angles at centre and circumference, angles of a cyclic quadrilateral. Tangent properties.

*The conics.* Ellipse, constructions and area. Constructions of parabola and hyperbola.

*Similarity.* Similar triangles; similar figures. Scales; maps and plans.

*Solids.* Properties, surfaces and volumes of: cuboid, cylinder, prism, cone, pyramid, sphere. Nets of solids.

*Loci and envelopes.* Simple examples.

*Symmetry.* Centre of symmetry, axes of symmetry.

Geometrical constructions, the use of instruments, length measure, etc., will be incidental to the above work.

## THE USE OF PRACTICAL SITUATIONS

Many instances of the practical use of geometrical ideas and constructions will be found in Chapters XII-XIV. In some cases a situation will be chosen because it illustrates a particular notion. For example, in the early work on angle measure it is important to emphasize an angle as a measure of change of direction or rotation. All too often, children mistakenly regard an angle as in some way measuring the space between two intersecting lines.

In other cases a situation may use a number of ideas and, perhaps, introduce fresh notions. In navigation there are many methods of fixing position, all of which use geometrical properties. Some examples are given below.

(*a*) In Fig. 28, *A* and *B* are two points on the coast, and their distance apart is known. The position of a ship, *S*, is fixed by measuring the angles *BAS*, *ABS*, and then making a scale drawing. The drawing requires the construction of a triangle given one side and two angles.

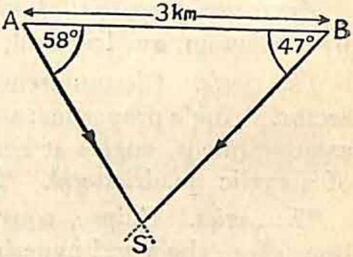


FIG. 28

The difficulty of the problem is increased if *A* and *B* are not intervisible, so that it is not possible to measure $\angle BAS$ and $\angle ABS$. The magnetic bearings of *S* from *A* and *B* are measured instead.

The speed of the ship and its track can be found by taking two sets of bearings with some interval of time between them. The time is noted when each set of observations is taken, and the time interval between them is found.

(*b*) A ship is sailing along the track *ABC* (Fig. 29). At *D* is a lighthouse. A note is made of the time when the ship is at *A*, where the angle between the track and the direction of the lighthouse is 45°. The time is again noted at *B*, where the angle is 90°. From these times and the speed of the ship, the distance *AB* can be found.

The triangle *ABD* is an isosceles right-angled triangle with $BD = AB$.

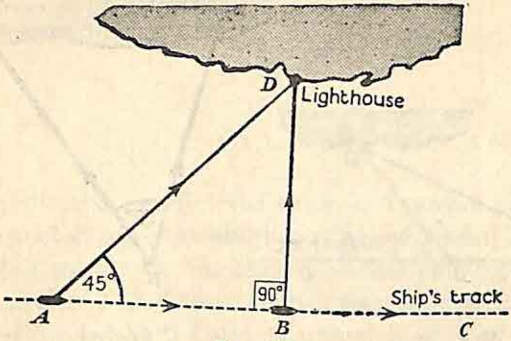


FIG. 29

(*c*) Here is another use of an isosceles triangle, and of the relationship between the exterior angle of a triangle and the interior angles.

The angle between the track and the direction of a landmark *D* is measured (position *A*, Fig. 30). The time at which this is done is noted. The time is again recorded when the angle is doubled (position *B*).

$\angle DBC = 2\angle DAB$

But $\angle DBC = \angle DAB + \angle ADB$ (exterior angle of triangle.)

$\therefore \angle DAB = \angle ADB$.

The triangle *ABD* is an isosceles triangle with $BD = AB$. The distance *AB* may be found from the speed of the ship and the time taken to travel from *A* to *B*. Hence the distance *BD* may be found.

(*d*) The following method can introduce the angle properties of a circle. From a ship *S*, the angle *ASB* between two landmarks is measured (Fig. 31). The distance *AB* is known. Can we then find the position of *S*? The following practical method is effective.

Mark out *AB* to a convenient scale and place pins at *A* and *B* (Fig. 32). Draw the measured angle on paper and

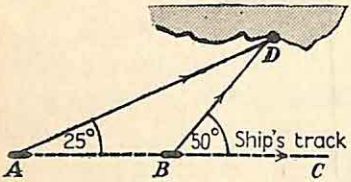


FIG. 30

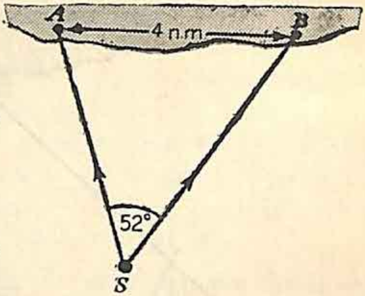


FIG. 31

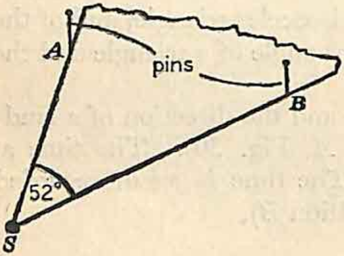


FIG. 32

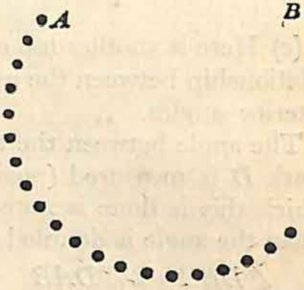


FIG. 33

cut it out. Place it so that the two arms touch the pins. Mark the point *S*. The paper angle can be moved between the pins and a number of positions for *S* are marked. It is soon seen that these points lie on the circumference of a circle (Fig. 33). The exercise can be repeated for acute angles, obtuse angles, and right angles, and the corresponding parts of circles noted. The situation clearly leads to a first investigation of the angle properties of circles.

When using this method in navigation two pairs of landmarks are observed. For each observation a circle may be drawn. The intersection of the two circles fixes the position of the observer.

# VIII

# ALGEBRA

As with all other studies the place and purpose of algebra in the school curriculum should be given careful consideration. The subject is not easy to define and many people confuse algebra with the symbolism which is used in algebra. For them algebra is the use of letters instead of numbers. This is mistaking a tool of the subject for the subject itself. Algebra has existed for over 3,000 years; symbols were first introduced about 500 years ago, and their use is still growing. Before their introduction all mathematical statements were written in the language of ordinary speech as Euclidean geometry is to this day.

When a child states that he can find the area of a rectangle by multiplying the length by the breadth, he has taken the first step in algebra. He is expressing a structured general relationship which he has observed. Symbolism reduces this statement to the concise and convenient form, $A = LB$. The first and most important step in introducing algebra is to give the children practice in observing relationships and expressing them. Throughout all the work done in algebra, emphasis should be placed on algebra as a form of language. The ability to state the mathematical content of a situation is one of the most valuable skills we can give to a child at the high school stage. The concise methods of expression of algebra are of particular value.

What can be done effectively at the high school level then? We can at least show the convenience of the algebraic language and its power in dealing with problems. If this is well done then the pupils will have received a first insight to the real quality of mathematical thought. Those who carry the study of the subject farther after they leave school will have been given a very sound understanding of the first ideas of algebra which will benefit considerably their later studies.

## PRELIMINARY PRACTICE

The importance of giving practice in expressing relationships has already been mentioned. Another useful preliminary practice is the solution of puzzles of the form, *I think of a number, I add 4, and the answer is 9. What is the number?* In finding the answers, the pupils are solving equations mentally. Practice of this kind might be started in standard VIII and should be continued regularly for at least five minutes once a week or fortnight. The work will be more interesting if the class help to compile the questions. One child is asked to think of a number, another to suggest an operation to be performed on it. The first child then gives the result, and the class then discover the unknown number. When some skill has been acquired in solving mental equations of this kind, which involve one operation, then the class can progress to equations involving two operations, for example, *I think of a number, add 2, and multiply by 3. The answer is 15. What is the number?* Some pupils will need guidance especially in the order of reversing operations. The last step was to multiply by 3; the first step in the solution is to divide by 3. This now leaves us with the statement, *I think of a number and add 2. The answer is 5.* The final step is obvious and easy.

## INTRODUCTION TO WRITTEN ALGEBRA

The most suitable introduction to written algebra is through the construction of formulae. A situation containing some simple mathematical relationship is chosen, and a few numerical examples are worked in order to emphasize the nature of the relationship. The relationship is then described verbally, and a formula obtained as a brief expression.

As an illustration, the class might be asked to work examples finding distance travelled when the speed of travel and the time taken are both known. After working a few numerical examples they write a sentence stating how to find the distance from the other two quantities. This is condensed to the brief statement,

$$\text{Distance} = \text{Speed} \times \text{Time},$$

and then to the formula,

$$D = S \times T.$$

The letters chosen to indicate the quantities should be suggestive in the early days, such as $D$ for Distance, etc.

Normally such a formula would be written

$$D = ST.$$

The multiplication sign is generally omitted in algebra. It is not possible, of course, to employ this economy in arithmetical expressions.

No attempt should be made at this stage at transposition in formulae. The distance-speed-time relationship should be treated as three separate formulae, $d = st$, $s = d/t$, $t = d/s$. Able pupils should see that these are three ways of stating the same relationship, so introducing the idea of transposition.

After some practice in constructing formulae, the pupils can be given formulae, and asked to express them in words, and to use them. The formulae must still represent situations which are understood easily. There is no need to spend much time on substitution in formulae; the practice given is in arithmetic rather than algebra.

## SOLUTION OF EQUATIONS

To make an early demonstration of the power of algebra, problems should be introduced as soon as possible. Before these can be tackled, it is necessary, however, to solve equations and also to deal with a certain amount of simple manipulation. The written solution of equations combines two skills already acquired: the mental solution of equations, and brief expression involving the use of symbols. One of the essential basic skills in solving problems is the translation of the problem into algebraic language, and it is well worth while to practice this skill in simple form. Some little time should be given to translating from words to equations, e.g.

| | |
|---|---|
| I think of a number, | $n$, |
| I add 3, | $n+3$ |
| and the answer is | $n+3=$ |
| 5 | $n+3=5$. |

The reverse operation, translating from symbols to words, can also be practised:

| | |
|---|---|
| $n$ | I think of a number, |
| $n-7$ | I take away 7, |
| $n-7=$ | and the answer is |
| $n-7=1$ | 1. |

Although the pupils will probably have had a large amount of practice in solving mental equations, the written solution generally provides some difficulties of expression. Their method of solving has undoubtedly been by reversal of operations: if 3 has been added to a number, then 3 must be subtracted from the result. It is possible that some are still not fully aware of the operations they carry out mentally. It should be emphasized that their method is a reversal of operations.

In the equation, $n+7=10$, able pupils should see that if we subtract 7, the left-hand side becomes $n$. For others the following argument and steps may be needed:

1. Translate the equation to words: 'I think of a number, I add 7 and the answer is 10.'
2. State the operation which has been carried out: 7 has been added.
3. State the operation needed to find the original number: 7 must be subtracted.
4. Write the left-hand side, subtracting 7: $n+7-7$.
5. We have subtracted 7 from the left-hand side. What must be done to make sure that both sides are still equal? 7 must be subtracted from the right-hand side.

$$n+7-7=10-7.$$

6. The solution is completed.

## SOLUTION OF PROBLEMS

As soon as some competence is acquired in solving equations they can be used in the solution of simple problems. It is not easy to find a range of problems which are simple enough and at the same time worth while. On the whole it is better not to spend too much time on this stage. After the first novelty of the method has worn off, the triviality of the problems becomes apparent. It should be sufficient to demonstrate the power of the subject by a few problems whose solution is not quickly seen by arithmetical methods. The following problem is probably beyond the arithmetical ability of most pupils. Its solution by algebra is quite simple.

A forest contained 450 sal trees and some deodar trees. 150 deodars were cut down and replaced by 125 young sals. Later, 160 sal trees were cut down and replaced by 180 young deodars. There were then twice as many deodar trees as sals. How many deodars were there at first?

Let the number of deodars at first be $x$.

| | *Sals* | *Deodars* | |
|---|---|---|---|
| At first | 450 | $x$ | |
| 125 sals planted | $450 + 125 = 575$ | $x - 150$ | 150 deodars cut |
| 160 sals cut | $575 - 160 = 415$ | $x - 150 + 180$<br>$= x + 30$ | 180 deodars planted |

$$\therefore x + 30 = 2 \times 415.$$
$$x + 30 = 830.$$
$$x = 800.$$

**There were 800 deodars at first.**

When equations involving two operations, such as *$2n + 3 = 11$*, are treated, emphasis must be given to reversing the operations one at a time. Again, the pupils should have been solving such equations mentally for some time. It is likely that they are more aware of the processes involved in two-step equations than in the case of single-step equations, on account of the greater difficulty of the former.

Once the written solution of these equations has been mastered, a wider field of problems is available, and more time may be spent on them. Problems can now be tackled which are quite difficult to solve by arithmetical methods, yet simple by algebraic means. The problems are necessarily more complicated, and it is advisable to give step-by-step practice in translating and solving at first. For example:

(*a*) *In a grove there are mango trees and neem trees. There are 20 more mango trees than neem trees. If the number of neem trees is $x$, how many mango trees are there?* ($x + 20$.)

(*b*) *How many trees are there altogether?* ($x + x + 20 = 2x + 20$.)

(*c*) *Altogether there are* 110 *trees. Write the equation which shows this.* ($2x + 20 = 110$.)

(*d*) *Solve the equation.* ($x = 45$.)

(*e*) *What does the solution tell us?* (There are 45 neem trees).

As the pupils gain skill, so the number of steps in a solution can be reduced.

## FUNCTION CONCEPT

Function concept in Algebra must be taught clearly. If it is not, it will cause pupils many difficulties later on. When I say that the distance I walk depends on my speed, distance is a function of speed.

The simple interest on a certain sum of money depends on the principal, rate and time and so simple interest is a function of P, r and n.

In fact every formula is a compact expression of a function concept. It is ' compressed information ' capable of being adapted for several particular cases of the type. Thus, when we say the volume of a right circular cone is given by the formula $V = \frac{1}{3}r^2h$, we know the interrelation between V, r and h in respect of the cone. For different values of r and h we can arrive at the corresponding volume.

Again the formula for the sum of the first n natural number is $S = \frac{n(n+1)}{2}$. This gives a compact method of finding the sum of any number of natural numbers starting with 1.

Thus $1 + 2 + 3 + \ldots\ldots + 100 = \frac{100 \times 101}{2} = 5050$.

## QUADRATIC EQUATION

If $P \times Q = 0$, then $P = 0$ or $Q = 0$.

This principle is used in the solution of a quadratic equation. Thus in $x^2 - 7x + 12 = 0$.

We can rewrite this as $(x-3)(x-4) = 0$.

$$\therefore \quad x - 3 = 0 \text{ or } x - 4 = 0.$$

$$\therefore \quad x = 3 \text{ or } 4.$$

Again, if $x^2 - 7 = 0$,

we have $x(x-7) = 0$ $\quad\therefore\quad x = 0$ or $x = 7$.

In general, if the equation is $ax^2 + bx + c = 0$, the roots are derived to be $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by the method of completing the square.

Bhaskara's method of arriving at this result is noteworthy.

$$ax^2 + bx = -c$$

Multiplying both sides by 4a

$$4a^2x^2 + 4abx = -4ac$$

Add $b^2$ to both sides

$$4a^2x^2 + 4abx + b^2 = -4ac + b^2$$

$$\therefore \quad 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This has, it will be noted, the advantage of avoiding fractions in the process of arriving at the result.

## GRAPHS

The graph of $y = mx$, and $y = mx + c$ may be taught and the significance of m and c explained by a study of several graphs.

It is essential to instil in the minds of pupils that the graph drawn is of the nature of a locus. Thus the graph of $y = 3x + 5$ is the locus of points of which the y co-ordinate and x co-ordinate satisfy the relation (or condition) $y = 3x + 5$. The nature of the graph of $y = k$ may be derived by making the pupils plot points where y remains constant and x co-ordinates may take any value. They will clearly see it is a line parallel to x axis.

Similarly $x = k$ is a line parallel to y axis.

This will lead to their understanding that the x axis is the graph of $y = 0$ and y axis the graph of $x = 0$.

### *Solution of simultaneous equations*

Suppose it is required to solve graphically

$$5x + y = 35 \text{ and } 2x - 3y + 20 = 0.$$

These can be rewritten in the form $y = -5x + 35$

$$\text{and } y = \frac{2x}{3} + \frac{20}{3}.$$

The graph of these may be drawn.

Since each graph is a locus, the intersection of these gives a point satisfying both conditions (or equations).

### *Solution of simple equations*

Take $7x + 3 = 0$.

Rewrite this as $y = 0$, $y = 7x + 3$.

Then note the point of intersection of the x axis (graph of $y = 0$) and the graph of $y = 7x + 3$. The x co-ordinate of the point of intersection is the required answer root.

*Solution of quadratic equations*

Suppose it is required to solve $2x^2 - 3x + 1 = 0$.

(1) We can write this as $y = 2x^2 - 3x + 1$.

$$y = 0.$$

The graph of $y = 2x^2 - 3x + 1$ may be drawn and the point of intersection of the parabola with x axis ($y = 0$) may be noted. The y co-ordinates of these points gives the two roots of the equation.

(2) This question can also be done by considering $2x^2 - 3x + 1 = 0$ differently, as follows:

$2x^2 = 3x - 1.$

This can be cast as $y = 2x^2$

$$y = 3x - 1.$$

The graphs of $y = 2x^2$ and $y = 3x - 1$ may be drawn. The straight line and the parabola will cut at two points. The x co-ordinates of these points will yield the required roots.

(3) There is a third method in which the graph of $y = x^2$ is drawn first whatever the equation is and then, by means of a suitable straight line graph, the roots are determined as the x co-ordinates of the points of intersection of the line and the parabola.

Let us take a few cases:

(1) $x^2 - 3x + 4 = 0.$

Rewritten, this becomes $x^2 = 3x - 4.$

This can be cast as $\left.\begin{aligned} y &= x^2 \\ y &= 3x - 4. \end{aligned}\right\}$

$\therefore$ Draw graphs of $y = x^2$ and $y = 3x - 4$.

(2) $2x^2 - 3x + 1 = 0$ becomes

$$2x^2 = 3x - 1 \qquad x^2 = \frac{3}{2}x - \frac{1}{2}.$$

This can be cast as $y = x^2$ and

$$y = \frac{3}{2}x - \frac{1}{2}.$$

$\therefore$ Draw graphs of $y = x^2$ and $y = \frac{3}{2}x - \frac{1}{2}$.

Thus it is seen that any quadratic equation can be solved graphically by drawing first the graph of $y = x^2$ and drawing in addition a suitable straight line graph, with a view to finding the point of intersection, the x co-ordinates of which are the required roots.

It is suggested that if a standard $y = x^2$ graph (point) is made available commercially (drawn, say, with the scale x axis 1 cm $\doteq$ 1 unit; y axis 0.4 cm = 1 unit), it remains only to draw a suitable extra line for each equation. Printed graph sheets with $y = x^2$ graph on it may be supplied to the boys. As an alternative the use of stencils may be considered for the same purpose.

## IX

## TRIGONOMETRY

SOME years ago trigonometry was regarded as an advanced mathematical topic only to be studied at the higher levels. There is now a growing tendency to introduce numerical trigonometry at a much earlier age, and its simpler parts are quite suitable for high school pupils. It has the advantage that it is a ' useful ' subject, and pupils quickly understand its merits.

The fundamental ideas of trigonometry have existed for over 3,000 years; there is evidence of them in the Rhind papyrus (1650 B.C.). These first ideas arose from practical problems dealing with heights, distances, and angles of elevation. The first trigonometrical table was compiled by Hipparchus (150 B.C.) to help him with his work in astronomy. The trigonometry of the high school should arise in the same way from practical work. A knowledge of the properties of similar triangles is needed; similarity should therefore be introduced or revised before the first work in trigonometry.

### STARTING TRIGONOMETRY

If the work is started early in the curriculum, only one ratio should be used in the introductory stages, and some time should elapse before dealing with any further ratios. If, however, trigonometry is not taught until the fourth form or standard IX, it may be necessary to introduce sine, cosine, and tangent together, or with only short intervals between them. The earlier start is strongly recommended. Historically the sine was the first ratio used. For high school pupils it is probably most suitable to start with the tangent ratio. It provides the simplest work and the most obvious situations.

An effective introduction is through heights of trees, buildings, etc. Pupils will undoubtedly have been finding heights through scale drawing some time before trigonometry is started. A set of drawings is made to find heights for various angles of elevation, using a base-line of 30 m. If possible, the observations should be real ones taken in the vicinity of the school, or playing-field. If the tallest objects are not very high, less than 18 m, for example, it is advisable to use a base-line of 15 m instead of 30 m. If the observations are not made practically, a regular set of angles can be used, 10°, 20°, 30°, etc. (Fig. 1). From the drawings a table of heights and angles of elevation is obtained. These values can be plotted on a graph (Fig. 2), and by interpolation the heights for other angles of elevation may be found.

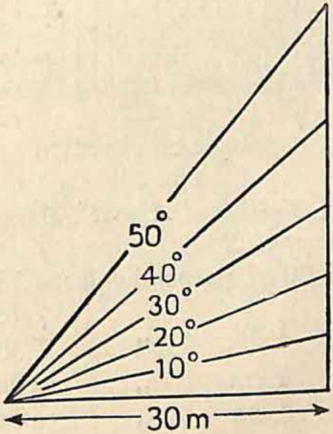


FIG. 1

By simple proportion, the heights for other base-lines can also be found, and it will be seen that the graph may be used for any right-angled triangle.

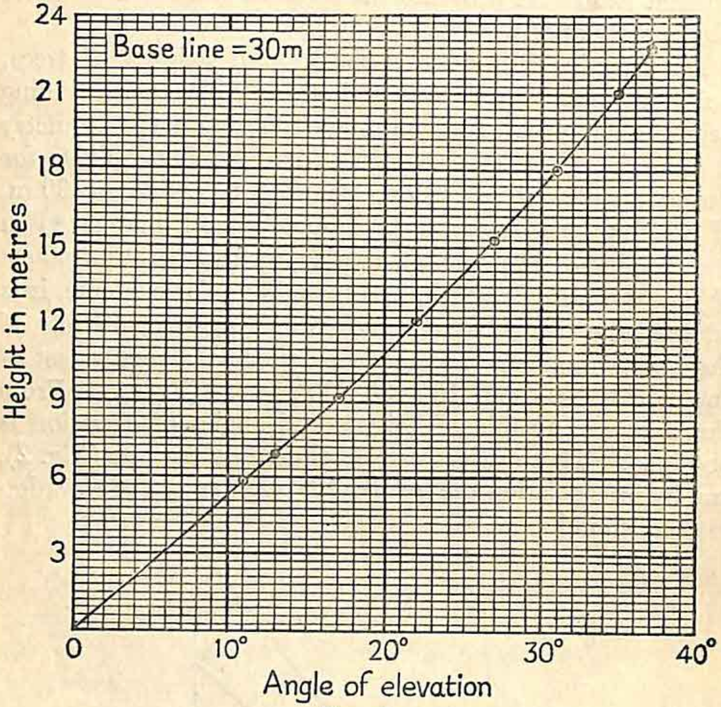


FIG. 2

*In a triangle,* $\angle B = 90°$, $\angle A = 31°$, $AB = 4.4\,cm$ (Fig. 3). *What is the length of BC?*

For a base of 30 m, perpendicular = 18 m (Fig. 2).

$\therefore$ ,, 1 m ,, $= \frac{18}{30} = 0{\cdot}6$ m

,, 1 cm ,, $= 0{\cdot}6$ cm

,, 4·4 cm ,, $= 0{\cdot}6 \text{ cm} \times 4{\cdot}4$

$= 2{\cdot}64$ cm

## THE TABLE OF TANGENTS

Once the value of the graph is seen in this way, the table of tangents can be introduced. (The graph already used is the same shape as the tangent graph.) There are two ways of regarding the tangent of an angle: as the ratio of two sides (*BC*/*AB* in the triangle of Fig. 3); or as a multiplier (to find *BC* we multiply *AB* by tan ∠*A*). The latter is recommended for early work in trigonometry.

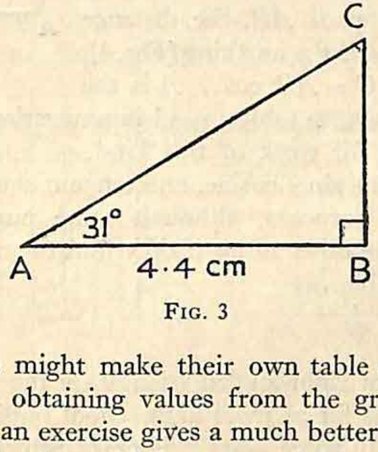


FIG. 3

The pupils might make their own table of tangents (to 2 figures) by obtaining values from the graph they have drawn. Such an exercise gives a much better understanding of the tangent table. If the range of values shown in the graph is not sufficient, further drawings can be made. It will not be possible to plot values approaching 90° because of the large values of the tangents. The pupils's table of tangents can be checked against a standard table.

## SINE AND COSINE

We have already stated that the next ratio should not be introduced for some time. The pupils will absorb the basic ideas of trigonometry more readily if they have only one ratio to deal with at first. Since the notion of a relationship between a side and angle has been thoroughly treated for the tangent ratio, the sine ratio does not need the same careful introduction. A practical situation should again be used; and a useful one lies in the northings, westings, etc., used in navigation. The same situation can be used for the cosine. If a vessel has sailed along the path $AB$, the distance $BC = AB \sin \angle A$ is the northing (Fig. 4). The distance $AC = AB \cos \angle A$ is the easting. The traverse tables, used in navigation, are useful ready reckoners for work of this kind.

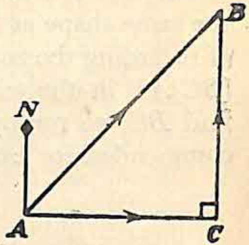


FIG. 4

The three ratios sine, cosine, and tangent should be sufficient for most purposes, although some pupils may be curious about the other ratios if they find them in books of mathematical tables.

## EXTENSIONS

The solution of right-angled triangles using sine, cosine, and tangent is as far as most high school pupils will go in trigonometry. In some cases it is permissible to use more advanced trigonometrical formulae in order to solve particular practical problems. For example, in a study of navigation a method of calculating great circle distance may be needed. Provided the problem is understood pupils can be allowed to use a formula without being able to derive the formula for themselves.

Some of these formulae entail finding the ratios of obtuse angles. They are found as follows:

Sine of an obtuse angle = sine of its supplement,
e.g. $\sin 120^\circ = \sin (180^\circ - 120^\circ) = \sin 60^\circ = 0{\cdot}866$

Cosine of an obtuse angle = − cosine of its supplement, e.g. cos 120° = − cos (180° − 120°) = −cos 60° = −0.5

Tangent of an obtuse angle = − tangent of its supplement, e.g. tan 120° = −tan (180° − 120°) = −tan 60° = −1·73

The following extensions might be made with able pupils:

1. *Trigonometrical graphs.* A sine graph can be a useful introduction to periodicity. The simple curve for values from 0° to 90° is first drawn. It is best to do this by means of a drawing construction. A quadrant of 10 cm radius is constructed and a set of radii are drawn at 10° intervals (Fig. 5). The perpendiculars dropped from the ends of these radii will give the values of the sines multiplied by 10. These can be projected across to give the points on a graph as shown. The vertical scale, 10 cm in length, represents values of sines from 0 to 1.

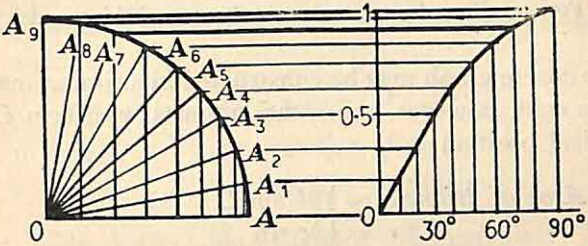


FIG. 5

For really able pupils the graph can be continued for angle values above 90° (Fig. 6). If the arm $OA$ is considered to go on rotating, then the angle swept out will increase through 90°, 180°, 270°, etc. Perpendiculars are dropped at 10° intervals as before, and projected across to give the points on a graph. Pupils who are able to follow this construction should also be able to see that, while we may not be able to draw an angle of more than 360°, we can turn through an angle greater than 360°. The arm $OA$ may be considered to go on rotating, and the angle of 370° will be seen to have the same sine as 10°, etc. The graph goes on repeating itself for each rotation of 360°.

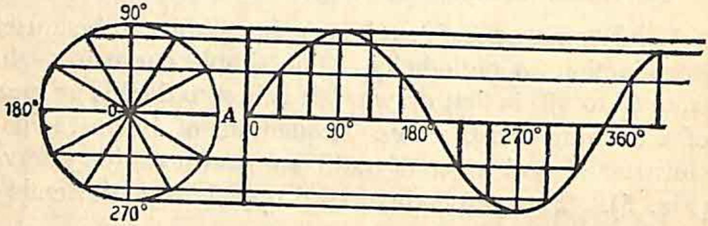


FIG. 6. (For clarity, angles are shown at 30° intervals)

The cosine graph may be constructed in a similar manner. In this case, however, the rotation starts with arm $OA$ in a vertical position.

2. *Area of triangle*

$$\begin{aligned} &= \tfrac{1}{2}ab \sin C \\ &= \tfrac{1}{2}bc \sin A \\ &= \tfrac{1}{2}ac \sin B. \end{aligned}$$

3. *Sine rule* (for solution of triangles given one side and two angles).

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

4. *Trigonometrical ratios of 30°, 45°, 60°.*

5. *Radians.* In certain cases measuring an angle in radians simplifies work considerably. A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle (Fig. 7).

$$2\pi \text{ radians} = 360^\circ, \text{ or } 1 \text{ radian} = 57^\circ \text{ approx.}$$

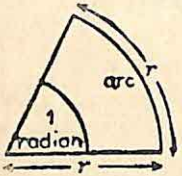


FIG. 7

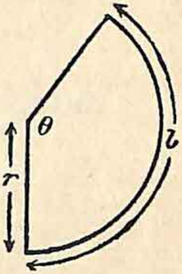


FIG. 8

This measure is of particular value when finding lengths of arcs, since:

$l = r\theta$, where $l$ = length of arc (Fig. 8)
$r$ = radius of circle
$\theta$ = angle, in radians.

Many table books, like Clarke's Mathematical Tables, include a table for converting degrees to radians.

The following example shows the use of radians in finding distances on the surface of the earth.

*Find the distance between Coimbatore (11°N, 77°E) and Chamrajnagar (11° 56′ N, 77°E).* Chamrajnagar lies due north of Coimbatore and their difference in latitude is 56′ (FIG. 9).

$56' = 0{\cdot}0163$ radians

Distance

$= 6{,}368 \text{ km} \times 0{\cdot}0163$
$= 104 \text{ km.}$

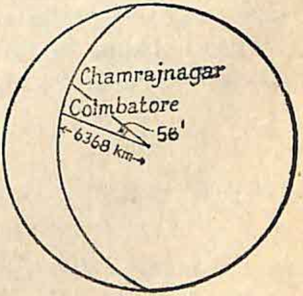


FIG. 9

# PART III

# *Work for the School*

## X

## TOPICS

THIS and the following chapters of Part III contain descriptions and examples of work suitable for Indian school pupils. The items are chosen for two reasons.

(*a*) They all show some purpose for the mathematics they use, although the purpose is not necessarily utilitarian. For example, a knowledge of how to draw the shape of the moon for any date is not an essential need of life, but it can satisfy the natural curiosity which most children possess.

(*b*) They are of a general character, and can be carried out in most schools.

The examples of topic work which follow are a few specimens only. There are hundreds of others equally valuable. The teacher should search for those which fit his needs and circumstances most closely.

### RADIO AND VISUAL EDUCATION (FILM SHOW) PROGRAMMES

This is a simple piece of topic work well suited to a class (and teacher) who have not done this kind of work before.

The organization is easy; the children are generally working together. When group work is done, the groups are carrying out similar tasks. The work is suitable for standards VI or VII and can be done by all but the least able children. The following processes are used: *addition and subtraction of number; graphical representation; tabulation; reading times (e.g. 8.40), addition and subtraction of time; percentages (not essential, but can be used).*

## THE POPULARITY POLL

It is always useful to attract the whole attention of a class from the very beginning of any piece of work. The question, 'Which is the best radio programme or visual education film show?', is sure to succeed in this instance. The discussions should be guided tactfully so that the first work to be done soon emerges. There is certain to be some argument and the names of several programmes will be brought forward. To avoid the argument becoming too heated, each child writes the names of the three programmes he likes most in order of preference. One child reads out his list, and a show of hands indicates how many other children have each item on their lists as well. The names of the obviously popular programmes are written on the blackboard together with the numbers of children listing each as one of their favourite programmes. This is continued, other children reading their lists until it is obvious that all the popular programmes are included.

From the list, half a dozen or so of the most popular programmes are chosen. Their degree of popularity can be decided in various ways. The simplest method is to find how many children listen to each programme. Another, rather more difficult, method is to ask each child to list the programmes in order of preference. Some kind of marking scheme is then adopted; e.g. if 6 programmes are placed in order, 6 marks are given to the first choice, 5 marks to the second choice, etc., down to 1 mark for the sixth position. Each programme is then taken in turn and the number of first choices, second choices, etc., noted. For this part of the work the class can be divided into groups, each group taking note of the details for one programme. The work will be made easier if, before the information is obtained, each child makes out a sheet of paper ready to enter the information as it is given. The total marks earned by each programme is then found. The children working in each group should act as a satisfactory check upon each other; those whose totals do not agree with that of the majority should discover where their errors occur. The errors can be due to faulty recording of information as well as mistakes in calculation; the former should always be checked first.

*Children's Hour*
1st × 6 =
2nd × 5 =
3rd × 4 =
4th × 3 =
5th × 2 =
6th × 1 =

Total

The results can be tabulated; they can also be shown in graphical form. The column type of diagram *(Fig. 1)* is probably the most suitable. The alternative form of diagram shown in Fig. 2 can be used to indicate what part of the class listens to each programme.

After the diagrams are drawn the children should write some description of the information they show. When children are not used to expression of this kind, discussion and questioning will be needed first of all to bring out the essential points. The written description, however, should always be the pupil's own effort. Copying sentences from a blackboard or from dictation may provide better grammar and phrasing, but it fails to give the child much needed and important practice in expression.

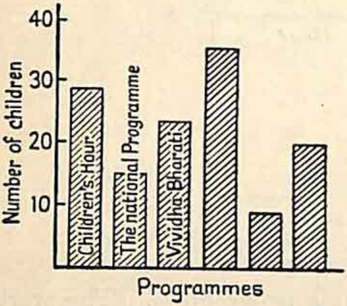


FIG. 1

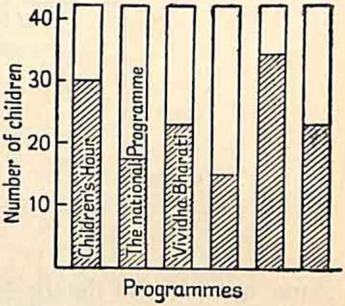


FIG. 2

NEWSPAPERS

This is a study of national or local newspapers. In most cases it will have to be a study of the national daily papers, but in some large towns a comparison of the local newspapers may be more interesting.

The topic can be started with the question, 'Which is the best newspaper?' Most children choose those taken in their own households. The numbers taking each paper can be found, tabulated, and shown in graphs. This information can be discovered from a single class or from the whole school.

Children are asked to bring old copies of the newspapers to school, and are grouped according to the newspaper they wish to study. Not less than four children should compose a group. The following information can be discovered about each paper, and comparisons made: *size of page, i.e., length, width, and area; number of pages, total area. Width of columns, number of columns per page, length of column, total length of columns per page. Thickness of paper used (fold to give 100 or 200 sheets; measure and divide). Weight of one paper. Price. Circulation (sometimes given, or can generally be obtained from publishers). Owner of newspaper, date of foundation.*

An estimate of the number of words in a newspaper can be made. This will clearly be a large number, and the pupils should first be asked to guess its size. These guesses should be written, not given orally, or the first guess will influence other pupils' estimates. The range of these guesses is sure to be interesting and should later be compared with the calculated estimate. The number of words in 20 lines is counted and the average number of words per line found. The number of lines per 5 cm is measured, and so the number of words per 5 cm of column is calculated. From this we can find approximately the number of words in the paper. It is obvious that the total column length of the paper must not be used, since not all the paper is occupied by close print. The total length of close print is not difficult to measure, and will use much addition of cm and parts of cm. There is no point in measuring each length very accurately; measurements to the nearest centimetre will suffice. The number of words in headlines, displayed advertisements, etc., may be estimated roughly; or a representative page can be taken and the number of words not in close print can be counted.

## THE CONTENTS OF THE NEWSPAPER

An analysis of the contents of the newspaper can follow. Discussion will fix the headings under which the analysis is to be made, e.g. general news, sports news, pictures, advertisements, correspondence, etc. A heading 'Miscellaneous' will be necessary, although this space can be found by subtracting the total found under other headings from the total for the paper. The groups find the amount of space given to each category in their newspaper, working in terms of area or column length. If area is used, it must be remembered that the total area of print is less than the total area of the paper, for allowance has to be made for the margin round each page.

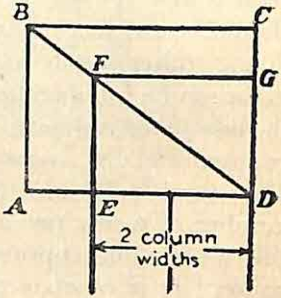


FIG. 3

It should also be noted that the printer works in terms of column widths. Pictures, displayed advertisements, etc., occupy 1, 2, 3, etc., column widths. Some work on proportion might be based on this. A given picture is to be inserted in the paper. Its width is to be reduced to two column widths. What will its height be? Having carried out a calculation of this kind, the class will no doubt be interested in the printer's method which involves no calculation (Fig. 3). *ABCD* is the size of the given picture. A diagonal *BD* is drawn. A distance *DE*, equal to two column widths, is marked off, on *AD*. *EF* is drawn perpendicular to *AD*, to cut the diagonal in *F*. *EF* is the required height, *EFGD* is the size of the reduced picture. This is a nice use of similarity, and may be employed in many other cases of reduction or enlargement.

The final analysis of each newspaper can be tabulated and shown in diagrams, percentages of space allotted to different subjects can be found, and the results from different papers may be compared.

The pupils might also be asked to look through their papers to see what use of number they can find. Numbers, both in figures and words, will certainly be discovered. Roman numerals are quite likely, also fractions, decimals, percentages, etc.

## LETTER DISTRIBUTION

An interesting extension is to use the papers for a study of the letters of the alphabet. Do we need the same number of each letter of the alphabet? Do we use as many a's as b's, as many c's as d's, etc.? Some pupils may see immediately that the vowel letters will be used more frequently than many others.

Each child takes a separate section containing about 500-1,000 letters. This will be about 15-25 lines of print, depending on the paper; the exact total is not important. He then records the number of times each letter is used. This is most easily done on a sheet of graph paper. The letters of the alphabet are placed along a horizontal line; a small graph square is filled by a stroke each time the letter is used (Fig. 4). If the word ' cab ' is read, a stroke is placed above C, another above A, and one above B. In this way the record builds up in graphical form. Despite the apparent

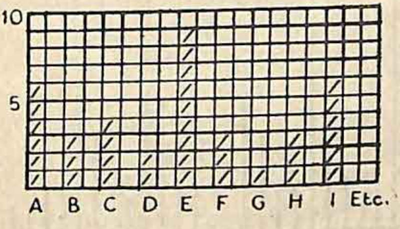


FIG. 4

monotony of this work most children find it completely fascinating.

When the records are completed they are compared. Even with so small a letter count some common features will begin to stand out, although a number of discrepancies will also be seen. The pupils then group in fours and combine their records, so giving a total of about 2,000-4,000 letters for each group. The diagrams are drawn and compared again. This is repeated forming larger and larger groups until the whole class is divided into two groups. At each grouping the diagrams are drawn and compared. They are best compared by using identical layouts and displaying them, with one vertically below the other. As the groups grow larger the graphs will generally be seen to resemble each other more closely. For totals of 10,000 letters or more, the results show a striking resemblance (Fig. 5).

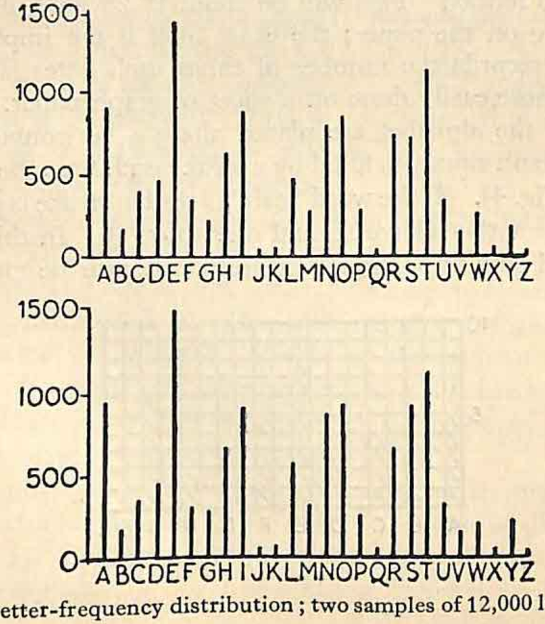


**Fig. 5.** Letter-frequency distribution; two samples of 12,000 letters each

This is an excellent simple study of statistical sampling. The information obtained about the frequency with which the letters are used is very much the same in each case, despite the fact that different samples were used. If any pupils in the class possess printing outfits, the numbers of letters supplied in a set might be compared with the results found from the papers. It is also interesting to write the letters in order of use, and by each write its symbol in the Morse code. A link will immediately be seen; the letters used most frequently possess the shortest symbols.

A letter count of the initial letters of words presents a variation; the distribution is not the same as before. The results might be compared to the number of pages or columns given to each letter in a dictionary or to any other large alphabetical list.

## MATHEMATICS IN THE HOME

The topics described so far may be dealt with either continuously, period after period, until the work is completed, or at intervals, only one or two periods being given to them each week. Now comes an example of a topic which should not be taken continuously. It deals with the shapes and sizes and quantities of materials used in the home. The possibilities are considerable, but as the information needed has to be searched for, there will be intervals while the pupils collect necessary data from home, from the local library, and elsewhere.

When dealing with flooring, for example, the pupils might measure one or more rooms in their own houses in order to find the number of tiles of a given size required for the floor. An examination of the problem will show that the width and length of each room are needed.

Discussion will often be necessary to indicate what information is required and in what form it is needed. The teacher should make no attempt to provide all the necessary information, although he may guide pupils in their search. Much of the value of this type of work lies in the training given in discovering needed information.

The following list outlines the possibilities.

### 1. *Structure of house*

(*a*) *Bricks.* Size and shape of bricks; methods of laying; drawings of different bonds (arrangement of bricks); number of bricks used in average house (from builder; or tables are available showing numbers of bricks used in different kinds of walling); cost of bricks for a house from the price per 100 or per 1,000; rate of laying bricks; bricklayer's wage-rates.

(*b*) *Wood.* Find the amount of structural timber, beams, joists, etc., in average house; roof structures, arrangement of beams, etc.; cost of structural timber. Doors and window frames, numbers and costs; measuring and drawing; glass, size of panes; total area; cost.

(*c*) *Roof.* Types of roof tiles; methods of laying; number required and cost.

(*d*) *Stairs.* Sizes of risers and treads; height of stairs, number of stairs. The amount of space which must be given on each floor to accommodate a flight of stairs can be solved by drawing. The headroom allowed must be at least about 2 metres.

### 2. *House and garden plans*

Reading plans, use of conventional drawing symbols; measuring own houses and drawing plans; improvements to own house plans. Floor space of house, cubic capacity of house.

Siting a house in a garden; plan of house and garden; garden plans, areas, etc.; size of vegetable plot needed to provide for family.

Building a work-shed, plans, materials needed, construction; garages, etc.

### 3. *Decorating and furnishing*

Areas of ceilings, walls, etc.; covering capacity of whitewash and paint; papering a room; cost of decorating. Curtains, width and length; floor coverings; arrangement of furniture; cost of furnishing.

### 4. *Running a home*

The number of items which can be dealt with under this heading is very large and it is probably better to select only a few. Some parents are reluctant to give details of food consumption, and matters of this kind must be treated tactfully.

Rent or building society repayments; rates; ground rent. Consumption of wood, charcoal, electricity, coal; weekly records from homes; costs per week, quarter, year. Cost of clothes per year; buying saris or dresses and making cholis, blouses, dresses, etc. Cost of papers and periodicals per week, month, year. Insurance. Saving for holidays. Catering for a religious ceremony or a birthday.

### 5. *Shapes in the home*

The home is an excellent source of plane and solid geometry. The children can be asked to look for examples of particular shapes and solids; rectangles, circles, ellipses, cubes, cuboids, cylinders, etc. A bucket may be recognized as the frustrum of a cone. The shapes might be drawn and described; from their descriptions should arise the general mathematical properties of the more common geometrical forms.

## POPULATION PROBLEMS

There are many population problems which can be studied with able pupils in Standards VII and VIII. For the country as a whole, most of the information required will be found from suitable reference books giving abstracts of statistics. *India—A Reference Annual*, is one such published by the Publications Division, Government of India. The following are extracts from *India—1971-72*.

1. *Growth of population*

| | 1951 | 1961 | 1971 |
|---|---|---|---|
| India | 36,09,50 365 | 43,90,72,582 | 54,73,67,926 |
| Andhra Pradesh | 3,11,15,259 | 3,59,83,447 | 4,33,94,951 |
| Gujarat | 1,62,62,657 | 2,06,33,350 | 2,66,87,186 |
| Kerala | 1,35,49,118 | 1,69,03,715 | 2,12,80,397 |
| Maharashtra | 3,20,02,564 | 3,95,53,718 | 5,03,35,492 |
| Mysore | 1,94,01,956 | 2,35,86,772 | 2,92,63,334 |
| Tamil Nadu | 3,01,19,047 | 3,36,86,953 | 4,11,03,125 |
| West Bengal | 2,62,99,980 | 3,49,26,279 | 4,44,40,095 |

2. *Changes in sex ratio*

| | Females for 1000 men | | |
|---|---|---|---|
| | 1951 | 1961 | 1971 |
| India | 946 | 941 | 932 |
| Andhra Pradesh | 986 | 981 | 977 |
| Gujarat | 952 | 940 | 936 |
| Kerala | 1,028 | 1,022 | 1,019 |
| Maharashtra | 941 | 936 | 932 |
| Mysore | 966 | 959 | 959 |
| Tamil Nadu | 1,007 | 992 | 979 |
| West Bengal | 865 | 878 | 892 |

### 3. *Population according to mother tongue (1961)*

| Languages (specified in Schedule VIII of the Constitution) | Persons speaking |
|---|---|
| Assamese | 68,03,460 |
| Bengali | 3,38,88,279 |
| Gujarati | 2,03,04,461 |
| Hindi | 13,34,33,207 |
| Kannada | 1,74,15,826 |
| Kashmiri | 19,56,114 |
| Malayalam | 1,70,15,674 |
| Marathi | 3,32,86,710 |
| Oriya | 1,57,19,387 |
| Punjabi | 1,09,50,660 |
| Sanskrit | 2,544 |
| Tamil | 3,05,62,671 |
| Telugu | 3,76,68,106 |
| Urdu | 2,33,23,399 |
| Other Indian Languages | 5,00,53,753 |
| Non-Indian Languages | 3 15,459 |
| Unclassified | 18,53,728 |

### 4. *Distribution according to age and sex (1961)*

| Age group | In thousands | |
|---|---|---|
| | Males | Females |
| Below 9 years | 66,240 | 64,494 |
| 10 — 14 years | 26,264 | 23,021 |
| 15 — 19 years | 18,590 | 17,276 |
| 20 — 24 years | 18,190 | 19,126 |
| 25 — 29 years | 18,525 | 18,043 |
| 30 — 34 years | 15,981 | 14,850 |
| 35 — 39 years | 13,599 | 11,857 |
| 40 — 44 years | 12,082 | 10,770 |
| 45 — 49 years | 9,733 | 8,320 |
| 50 — 54 years | 9,128 | 7,977 |
| 55 — 59 years | 5,281 | 4,548 |
| 60 — 64 years | 5,705 | 5,531 |
| 65 — 69 years | 2,472 | 2,378 |
| 70 and over | 4,176 | 4,442 |
| Age not stated | 95 | 81 |
| Total | 2,26,061 | 2,12,714 |

### 5. *Literary in India (1971)*

| States | Literates | | |
|---|---|---|---|
| | Persons | Males | Females |
| India | 160,610,191 | 111,911,000 | 48,699,191 |
| Andhra Pradesh | 10,657,130 | 7,299,949 | 3,357,181 |
| Gujarat | 9,533,847 | 6,364,992 | 3,168,855 |
| Kerala | 12,802,821 | 70,12,792 | 5,790,029 |
| Maharashtra | 19,670,142 | 13,365,266 | 6,304,876 |
| Mysore | 9,230 140 | 6,256,109 | 2,974,031 |
| Tamil Nadu | 16,189,712 | 10,734,758 | 5,454,954 |
| Uttar Pradesh | 19,123,468 | 14,903,166 | 4,220,302 |
| West Bengal | 14,688,745 | 10,063,468 | 4,625,277 |

### 6. *Birth and death rates (1951-1960)*

| Zone | States covered | Estimated | |
|---|---|---|---|
| | | Birth rate per cent | Death rate per cent |
| Central | Uttar Pradesh and Madhya Pradesh | 42.0 | 24.4 |
| Eastern | Assam, Bihar, Orissa and West Bengal | 43.3 | 23.9 |
| Northern | Punjab and Rajasthan | 43.6 | 19.0 |
| Southern | Andhra Pradesh, Kerala, Mysore and Tamil Nadu | 38.5 | 22.3 |
| Western | Gujarat and Maharashtra | 42.8 | 21.4 |

### 7. *Life expectancy (1931-1960)*

| Years | Expectation of life at birth | |
|---|---|---|
| | Males | Females |
| 1931 — 40 | 32.09 | 31.37 |
| 1941 — 50 | 32.45 | 31.66 |
| 1951 — 60 | 41.90 | 40.60 |

With these tables and more tables like these, various studies can be undertaken—the growth of the population, as a whole and sex-wise, language distribution, expectation of life, infant mortality, etc.

To mention a few of the problems,

1. the increase per cent in population from 1941 to 1951, 1951 to 1961 and 1941 to 1961 can be found out;
2. the density of population for India and for the States can be found;
3. the percentage of literacy as a whole and sex-wise for India and for the States can be found out.

Below is given a list of some topics for study with a brief outline for each.

*The coal industry*. Analyses of production costs, wholesalers' costs, and retailers' costs, showing how much of the retail cost of coal is consumed in transport; an explanation of the various costs. Annual coal production and consumption; analysis of consumption in lakhs of tons, and in percentages. Coal imports. Miners' wages; accidents in coal-mines.

*National finance*. Explanation of Budget. Analysis and comparison of two consecutive Budgets. Expenditure per head of population. Income-tax rates; death duties; customs duties; entertainment tax. The National Debt. Cost of Defence.

*Electricity*. Electrical units; volt, ampere, ohm, watt, kilowatt, kilowatt-hour. Exercises on consumption. Reading meters. Methods of charging, electricity bills. Annual outputs of electricity in India and other countries.

*Transport*. Reading railway time-tables; calculating average speeds. Using time-tables for cross-country journeys with changes of train. Fares. Graphs of train journeys; trains travelling in opposite directions; in the same direction at different speeds. Annual and weekly consumptions of coal. The growth of railways in the nineteenth century.

Reading bus time-tables; speeds; fares. Cost of making a new by-pass road; description of road, area, volume of concrete, cost per kilometre. Bridges, costs and data.

Air time-tables; the 24-hour clock, Greenwich Meridian time, fares, cost per kilometres. Air records.

*Personal finance.* P.O. savings bank; savings certificates; interest; statements; comparison of investments.

Buying a house; mortgages; repayments, weekly and annual; total costs. Rent.

House insurance; Life and other assurances.

H.P. charges; moneylenders' and pawnbrokers' charges.

Dividends.

*Kerala.* Weather; rainfall, monthly and annual, records of heavy rainfall and of drought; monthly temperatures, averages, record temperatures; cloud cover.

Health; birth rate and death rate; infectious diseases, cases and deaths.

Population; growth of population; employment; industries.

Imports and exports.

Rates; rateable value; analysis of expenditure. Education, schools, children, teachers, costs. Electricity supplies; water supply, capacity of reservoirs, consumption. Housing; houses built by corporation and by private builders.

Politics; local and parliamentary election results.

*Logarithms and the slide rule.* History of logarithms, advantages. Theory of indices. Graph of logarithms. Use of tables; multiplication and division; positive and negative characteristics. Calculations using logarithms. Squares and square roots.

Construction of logarithmic scales. Using the slide rule for multiplication and division. Use of professional slide rule. Squares, roots, etc. Construction of 100-cm slide rule.

*Scales and scale drawing*. Use of scales. Dividing a line into a number of equal parts; construction of plain scales. Construction of diagonal scale to measure metres and centimetres to a scale of 1 cm = 50 cm; construction of diagonal scale to measure 0·25 mm. Drawing to scale; solution of problems by scale drawing. Enlargement of map, etc., by method of squares (page 248).

*Development of solids*. Cubes, cuboids, etc.; drawing and calculating surface areas. Triangular and hexagonal prisms. Cylinder, development and area. Cone, development and area. Pyramid, development and area. Surface area of sphere.

*Air navigation*. Surface of the earth; great circle and small circle. Latitude and longitude; fixing position; degrees, minutes and seconds; nautical miles; change of latitude, change of longitude. Direction on the earth; true, magnetic, and compass bearings.

*Sea navigation*. Latitude and longitude; fixing position; longitude and time, finding longitude from local time. Latitude and the Pole Star. Nautical miles; distance of horizon. Distances along a meridian; distances along a parallel of latitude, using trigonometry (page 135). Solution of spherical triangle by formula.

*Small measuring*. Diagonal scales measuring to 0·25 mm. Micrometer screw gauge, construction and use. Construction of paper vernier scale reading to 0·25 mm. Using vernier calipers.

*Maps and mapmaking*. Conventional map symbols; scales. Change of scale; enlargement and reduction using pantograph (page 248) and method of squares. Metric and English measures. Contours, sections, profiles, gradients; exercises on intervisibility using drawing and calculation (page 203). Bearing compass, magnetic bearings, variation, true bearings.

*Surveying.* Survey of area in school yard using survey line and offsets; calculation of area. Survey of area measuring sides and angles; use of sine and cosine formulae; area of triangle by trigonometry. Width of river by drawing and by calculation. Traverse around the school block. Area of irregular figures using Simpson's rule. Levelling a field; gradient of regular slope, angle of slope.

*Trigonometry.* Scale drawings of heights and distances, graphs of heights and distances. Right-angled triangle, Pythagoras' theorem. Sine, cosine, and tangent; complementary angles. Calculation of heights, shadows, etc., using tangent ratio. Tangent graph. Problems involving sine and cosine; graphs. Practical fieldwork; calculating distances from measured base-line and angles; calculation of heights. Inaccessible heights and distances.

# XI

# THE MONEY FAIR: AN ACTIVITY

THIS activity is an extension of the Infant school 'shop', but it can be adapted to stretch the most able pupils. All usages of money, and much else, may be taught through the Money Fair in a lively and interesting manner. It is one of the most satisfactory forms of preparation for the economics of everyday life.

A number of shops and offices are set up in the classroom, and provided with all necessary material and information. Some children are chosen to act as shopkeepers or clerks; the rest of the class become their customers. These are given work-cards which detail tasks to be carried out at one or more of the shops or offices.

## SOURCE OF MONEY

Some source of money for the customer is necessary. The amount needed by a class is large and imitation money will probably have to be used; few teachers could finance the activity from their own pockets. The money can be given out with the work-cards, but this takes up too much of the teacher's time; other methods are to be preferred. The teacher should play as small a part as possible within the organization, so that he can be free to help with individual difficulties.

One simple method of providing the customers with money is to have a bank to which the customer goes as soon as he gets his work-card. On the back of the card is a statement such as *Pay bearer Rs 10*. The customer shows this to the bank clerk who then gives him the money. This method is suited to the lower streams and age groups. The bank may be a section of the Post Office so that the clerk fulfils other duties besides giving out money.

A better method, but one which demands more from the pupils, is to provide each customer with a P.O. savings bank account. When the activity is first started a sum of money is placed to the credit of each customer and shown in a record book kept in the Post Office. It is also shown in a bank book given to the customer. These bank books can be simple duplicated forms as shown below. When this system is first used the initial directions on each work-card should be, *Go to the Post Office, and draw out Rs 2* (or other amount). The customer obeys the direction; he fills in a withdrawal form, and the amount drawn out is entered on his book, and in the Post Office record, by the clerk. At a later stage such directions are omitted, and when they require money the customers are expected to estimate the amount of money they will need. They then go to the Post Office to draw out a sum sufficient for their needs.

POST OFFICE SAVINGS BANK

NAME................ No................

| DEPOSITS | Rs | p | WITHDRAWALS | Rs | p |
|---|---|---|---|---|---|
| | | | | | |

Yet another method of providing the customers with money is to give them a current account in a bank. This can be done with the more able pupils. Many transactions will then be carried out using cheques. They should use both Post Office savings bank and current accounts together.

As the work proceeds the money is spent and there comes a time when the customer finds he has not enough in his accounts to meet his needs. To counter this difficulty, a few work-cards are made for the special purpose of replenishing low accounts. Devices such as the 'sale of a car', 'sale of a house', or 'sale of a crop of paddy' may be used, and the money received can arrive in the form of a cheque made payable to bearer. A large number of savings certificates can be given, to be cashed and the money divided between current account and savings bank.

### SHOPS AND OFFICES

The nature of the shops and offices in the Money Fair will depend on the class. In regard to pupils of low ability, the activity would concentrate on simple shopping, together with a Post Office. With better pupils in the higher classes the following can be used: Bank, Post Office, Insurance Office, including Building Society facilities for house purchase, Municipal Services (water, electricity, rates), Multiple Store (to cover all aspects of shopping), Travel Agency.

The number of shops used depends upon the size of the class; five or six shops will be found suitable for a class of forty children. A useful ratio, which is helpful in getting the activity started, is 1 shop with 2 servers for every 6 customers. These numbers can be adjusted as the work proceeds. It will be found that the tasks of some offices, e.g., the Travel Agency, often take some time and it may be necessary to employ three clerks. In others one clerk or server might be sufficient. The more intelligent pupils should be chosen to act as clerks for much of the organization depends upon them. They should not, however, be kept in one shop for more than a few weeks. They can be moved from shop to shop, or they can change places with some of the customers. The changes should be made gradually, moving not more than one at a time from each shop, so that there is always someone in charge who has become accustomed to the routine of the shop.

It is important to ensure that the shop assistants or office clerks do not do too much of the work. Their duty is to provide the customers with all necessary information and to act as a check on the work. For example, when details of a train journey are required, the clerk provides the necessary time-tables and gives any help needed, but it is the customer who carries out the actual work. When the work has been completed the customer should write a description of the transaction, showing any necessary calculations. This written work is collected and marked in the normal manner by the teacher.

The shops and offices need not be elaborate; all effort should be directed to supplying information, materials, etc., to provide as wide a range of topics as possible. In younger classes the shops can display goods and prices on posters. At a later stage, indexed catalogues and leaflets can be used. These may be obtained from firms selling furniture, gardening equipment, cycles, radio sets, etc. If they are not already indexed a brief index should be made. This gives useful practice in the use of an index; it also saves time in finding the required articles. When, in the later years, the separate shops become the departments of a multiple store, separate catalogues are better than a complete catalogue as issued by some stores. Only one customer at a time can use a catalogue, so a single large catalogue will cause delays.

With a little rearrangement of furniture, the activity can be conducted in a class-room without difficulty, although a less confined and crowded space has advantages. The shops, etc., are arranged in the corners and at the front of the room. Ample writing space must be provided for the customers. In some cases it may be convenient to place one or two shops outside the class-room, in the corridor.

The Post Office and Bank are the only offices which require a large amount of money; the others collect money from the customers. They can be given two ten-rupee notes to start with; if change is required, one of the assistants goes to the bank to change a ten-rupee note. The shops and offices should keep accounts, making lists of money taken and paid out, and checking against the cash in hand. In later years, when the bank is used, all the shops and offices should have bank accounts, and pay in the cash and cheques received.

## RECORD OF WORK

A record must be kept of the work-cards completed by each pupil. The work-cards should be classified; the most satisfactory method is to label those using the Post Office A1, A2, A3, etc.; those using the Travel Agency B1, B2, B3,

| | A1 | A2 | A3 | A4 | A5 |
|---|---|---|---|---|---|
| Abraham | / | | | × | |
| Benjamin | × | / | | | |
| Charles | | | × | | × |
| David | | × | | / | |

etc.; and so on. The surplus work-cards can either be kept at a central 'Employment Exchange', or in the shops and offices. In the former case, the pupils return the completed work-cards to the Exchange and obtain others. A register is kept showing the names of pupils and the numbers of the work-cards. A stroke, /, indicates a work-card given out; it is crossed off, ×, when the work-card is returned.

Alternatively the work-cards are kept in the shops or offices with which they are concerned. When a pupil wants a fresh work-card he collects one from any office or shop. Class lists are kept in each shop as in the Employment Exchange, and a record kept in the same way.

It is impossible to cover all the details of every kind of shop or office that can be used in such an organization. The outlines given below should give sufficient information to enable teachers to devise shops, etc., to suit their own needs.

## Post Office

Sale of stamps, postal orders, money orders, licences, savings certificates. Posting letters and parcels (inland, overseas, airmail, registered). Telegrams. Telephone charges and accounts. Savings bank accounts, interest, table of interest on savings certificates.

*Materials needed.* Duplicated forms for savings books, withdrawal of savings, savings certificates, postal orders, money orders, licences, telegrams, applications for licences, money orders, savings certificates, etc. Savings bank record book.

An information booklet would give details of interest rates, postal charges, etc., and would show how to make out licences and other forms, etc.

### INSURANCE OFFICE AND BUILDING SOCIETY

Insurances of various kinds; life, endowment, house, house contents, car, motor-cycle. Insurance claims. Buying houses; mortgages, deposits, repayments.

*Materials needed.* Leaflets giving information about insurance policies, premiums, etc.; also house purchase. (These are readily obtainable from Insurance Offices and Building Societies.) Duplicated forms for policies, applications for policies, claims, applications for loans for house purchase.

An information booklet would give simple descriptions of the types of insurance and explain the terms used (e.g. premium). Examples of completed policy forms, claim forms, etc., could be shown. A simple account of house purchase through a Building Society mortgage would also be included.

### MULTIPLE STORE

Buying goods of all kinds. A wide range of related tasks can arise here. A garden can be planned with help from the gardening department, and the plants, seeds, etc., purchased. The cost of a fitted carpet for a given room can be found with the help of the furnishing department.

Credit sales account, instalments payments; discount for earlier payments, interests charged.

*Materials needed.* Catalogues of various kinds, receipt forms. Information booklet.

### MUNICIPAL SERVICES

Paying rates, electricity bills. Reading meters; consumption of electricity. Buying electrical equipment; credit sales accounts.

*Materials needed.* Scales of electricity charges. Catalogues of electrical equipment. Duplicated forms for rate demands, bills, receipts. Information booklet.

### BANK

Using cheques, paying-in slips. Balance sheets. Loans. Foreign exchange, traveller's cheques.

*Materials needed.* Bank record book. Duplicated forms for cheques, applications for cheques, paying-in slips, foreign money, traveller's cheques. Information booklet.

### STARTING THE MONEY FAIR

It will be clear that the organization of the activity must be careful and detailed, and much thought must be given to finding a suitable method of introducing it to a class. Probably the most satisfactory method is to start with one office or shop at first, and then gradually expand with the help of the class. If this is done, the Post Office is the most suitable to commence with. The teacher first prepares all the material, work-cards, etc., for the Post Office, and selects three or four clerks. These are given time to examine the material, look at the work-cards, etc., and make sure that they understand their duties. This should take two or three periods spread over a week. When the ' clerks ' are ready a group of pupils, about 8-12 ' customers ', can use the Post Office while the rest of the class carry on with other work.

The activity is then expanded by introducing one new shop or office at a time. The pupils gain considerably by helping in the preparation of each new shop. A class discussion will often bring useful suggestions from the pupils.

They can help to collect and arrange material and assist in compiling work-cards. The latter will need careful scrutiny.

Once the activity has been introduced, some of the pupils with experience as ' clerks ', etc., can temporarily continue these functions when it is being introduced to another class. Where a body of trained assistants is available all the shops and offices can be introduced together. The basic material can be used for more than one class, but it is desirable to have separate sets of work-cards.

## THE MONEY FAIR AND THE SYLLABUS

The amount of time devoted to an activity such as the Money Fair will depend on the teacher's estimate of its value. If it is well organized so that the setting out and clearing away of the material are carried out promptly it can be operated for a single period at a time. The greatest benefit is probably derived from the activity by running it for a single period each week and linking it with the work done in some of the other mathematics lessons. Once a class has become accustomed to the operation of the Money Fair it is important to ensure that progress is made by introducing new ideas, new usages, and fresh demands on their arithmetical skills. It is all too easy to concentrate solely on the organization of the activity and thereby lose many opportunities for bringing in fresh work.

A study of the mathematics syllabus will show which parts of it can be dealt with by the Money Fair. A scheme of work can then be made for the Money Fair, detailing the processes, notions, etc., which it can use. The work-cards must be based on this scheme. A series of parallel formal lessons is often profitable. The Money Fair provides the situation which demands some new technique; the formal lesson emphasizes its mathematical nature and gives graded practice in its operation.

## ADAPTATION TO WEAKER PUPILS

The description given so far is intended to show the wide range of operations which can be carried out with able pupils. The work which can be done with other children is restricted according to their abilities, but is, nevertheless, valuable. With very backward children, the reading of the work-cards may present some difficulty. The wording and instructions must be kept as simple as possible. It has been found that the activity provides a very real incentive to read, and that reading is appreciably improved.

The work will be largely confined to shopping, and the work-cards could be as follows:

*Go to the Bank. Ask for Rs 2.*
*Go to Mr Roti. Buy some chapattis and sagu.*
*Write the sum which shows how much you have spent.*
*How much change have you?*
*Go to the Bank. Give in your change.*

To help the poor reader the layout of the work-cards and the sentences used should be standardized as far as possible so that the child encounters the same phrases many times. Coloured work-cards can also help. For example, all the work-cards related to the rotiwala's shop might be coloured yellow, while it advertises its goods on a yellow poster. Other colours are used for other shops, thus enabling the child to find easily the shop to which he must go.

Some variations from shopping are possible. On some mornings a ' bus ' might be operated by one child. The others take the bus to do their shopping, and buy tickets from the conductor. Simple bus time-tables may be used. If a 'railway' is preferred, tickets can be bought at a ticket office, and journeys can be planned from a time-table. The ticket office can also sell cinema and theatre tickets.

A 'hotel' needs little more than a menu showing a list of dishes and prices.

# XII

# SURVEYING AND MAPWORK

SURVEYING may be taken as a course lasting for four years; or courses of shorter length can begin at any time, or an isolated surveying topic lasting one or two lessons may be studied. Work can be found to suit all pupils, from the slowest to the most able. Surveying is more fitted for boys' schools where remarkably high standards are sometimes achieved.

Care should be taken to ensure that surveying is used as a medium for teaching mathematics, and that the mathematical principles are fully brought out. Many situations will be found for showing both arithmetic and geometry in use. With able pupils trigonometry may be freely used towards the end of the course.

In the following brief summary of surveying the basic skills of measuring distances, measuring angles, etc., are dealt with first, followed by a selection of exercises which can be carried out. Finally, notes on surveying instruments are given on page 191; no attempt is made to explain these in earlier references to them. It must be stressed that the order shown is not a teaching order; it is an attempt to convey the necessary information as concisely as possible.

## MEASUREMENT OF DISTANCE

Measuring to the nearest foot or link is sufficiently accurate for most outdoor work; smaller distances will probably not show on a map. The same distance measured two or three times, or by different groups, will demonstrate the degree of accuracy in measurement to be expected. The standard should, of course, improve with practice.

Distances can be measured by:

(*a*) *Chain.* The surveyor's chain is 66 feet (20·1168 metres) long, divided into 100 links. It is useful for land measure, especially area in acres. The engineer's chain, 100 feet (30·48 metres) long, measures in feet. It is too heavy for ordinary school use. Although we have now changed to the metric system in India, the 66-ft. chain continues to be very widely used.

(*b*) *Tape.* This is 66 feet or 100 feet long, generally subdivided in both inches and centimetres. Some 66-ft. tapes are divided into links on the reverse side. These are useful for land measure, or for use with a surveyor's chain.

(*c*) *String.* It is unlikely that sufficient chains or tapes will be available to provide for a whole class. They may be supplemented by lengths of string or wire. 66-ft. lengths knotted or marked at 6-ft. or 10-link (2·01168 metre) intervals will serve for measuring the longer distances. Distances less than 6 ft. can be estimated, or measured with a 6-ft. length knotted at 1-ft. intervals. Wire is less easy to handle than string, but gives more accurate results. String possesses a certain elasticity which allows it to stretch and so affect readings.

(*d*) *Pacing.* With practice this method can give a surprising degree of accuracy. It is also a useful source of examples dealing with multiplication and division of length. The length of one pace must first be found. This is best done by measuring the distance walked in 10 or 12 paces. This should be repeated three or four times to ensure that no errors occur. The normal walking stride should be used and not the longest pace of which the child is capable.

To check the children's using of pacing, the teacher can mark out a distance of 20 or 30 metres, not telling the class what the distance is. The pupils then find the distance by pacing, and report the results to the teacher. The pace should be rechecked at intervals of not more than a year.

(*e*) *Other methods.* For short distances, an alternative to pacing is to use the length of the pupil's foot, and measure in ' foot-lengths '. These are later converted to conventional measure.

A bicycle wheel, or any other fairly large wheel, may be used for making a trundle, or perambulator. This old surveying instrument measures distance by counting the rotations of a wheel as it is pushed along. Some device which gives a click at each turn of the wheel will help the counting of revolutions.

Some methods of measuring distances indirectly are given on page 174.

## LAYING OUT A STRAIGHT LINE

It is often necessary to measure the distance between two points which are more than one tape-length apart. It is important that the tape is set out so that it lies along the straight line between the two points. The points should be marked by ranging rods. One person remains at the first point with the beginning of the tape, while another walks towards the second rod, laying out the tape as he goes. It is most probable that he will be off the straight line, and he can then be correctly lined up by the first pupil who looks along the line of sight between the two rods. It is undesirable to attempt to pull the tape or chain into the straight line. A series of ' waves ' should be sent along it, at the same time pulling it in the required direction. When the tape is correctly laid out, an arrow is placed in the ground to indicate the end of the tape. The tape is then taken forward by its two bearers until its beginning coincides with the arrow. The process of lining up, etc., is then repeated.

When long distances are to be measured, the leading bearer should start off with ten arrows. The following bearer picks up each arrow from the ground as he moves forward. The number of arrows he has at the end of the exercise gives the number of times the tape has been set out.

Such a method of ensuring measurement in a straight line is not possible when using pacing or a trundle. If it is desired to measure the distance *AB* (Fig. 1) by pacing, a third rod, *C*, is positioned so that it lies on the extension of *AB*. This is done with the help of an observer at *A*. The measuring is then done from *A* to *B*. If, at any time, *C* becomes visible, it shows that the measurer has wandered from the straight line *AB*.

FIG. 1

MEASUREMENT OF ANGLES

(*a*) *Horizontal angles* may be measured with the anglemeter, bearing or sighting compass, theodolite, or sextant. Instruments such as anglemeters, theodolites, etc. which possess a fixed scale, should read from 0–360° in a clockwise direction. The angle between the directions of two points may be measured with these instruments by either of the following methods:

(i) Set the instrument scale so that one direction reads 0° on the scale. The required angle is found from the reading of the other direction (Fig. 2).

(ii) Take the readings of the two directions without setting the instrument scale (Fig. 3). The required angle is found by subtraction. This method is generally slightly more accurate, but rather more difficult for some children to understand since the angle is not obtained by measuring from a zero reading.

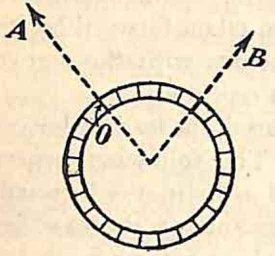


FIG. 2

Angles may be *fixed* (not measured) by tie lines, i.e. $\angle AOB$ can be fixed by measuring $OC$, $OD$, and the tie line $CD$ (Fig. 4). $C$ and $D$ can be any points on the lines $OA$, $OB$. The angle can be reproduced by drawing the triangle $OCD$ to scale, but its value remains unknown.

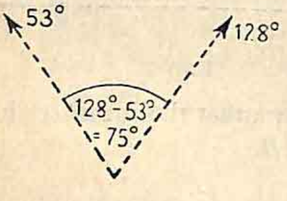


FIG. 3

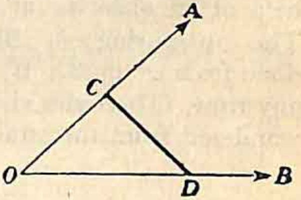


FIG. 4

(*b*) *Vertical angles* can be measured with the astrolabe, clinometer, Jacob's staff, theodolite, or sextant. In most instances the pupils will be measuring angles of elevation, and the instrument must possess some device which gives a zero horizontal reading. In the theodolite and sextant a spirit-level is used; the clinometer and the astrolabe make use of gravity. The construction of the latter pair is worth special attention since they use the relationship between horizontal and vertical directions. The Jacob's staff was commonly used in navigation for measuring the elevation of the sun, but in this case, the lower sight was fixed on the horizon.

(*c*) *Rough measure.* Instruments such as anglemeters and astrolabes are especially useful in the first years of the secondary school, since the movement of their pointers emphasizes the notion of angle as a measure of rotation. At a later stage the use of the following means of estimating angles might be introduced.

| | |
|---|---|
| Width of first finger when held at arm's length | 1° |
| Distance between first and second knuckles at arm's length | 2½° |
| Distance ,, ,, fourth ,, length | 8° |
| Width of palm at arm's length | 10° |
| ,, span ,, ,, | 20° |

(*d*) *Right angles.* In many surveying exercises it is necessary to set out one line perpendicular to another. The following methods may be used:

(i) Surveyor's cross. This is probably the simplest and quickest method, but the methods given below demonstrate various geometrical properties and are to be preferred.

(ii) 3 : 4 : 5 String triangle. This is a very simple device using Pythagoras' theorem. A 5 : 12 : 13 triangle is equally useful.

(iii) Swinging the tape (Fig. 5). A tape *AB* is fixed at *A*, and the other end swings across the base line. The position giving a minimum reading on the tape is the foot of the required perpendicular. This uses the fact

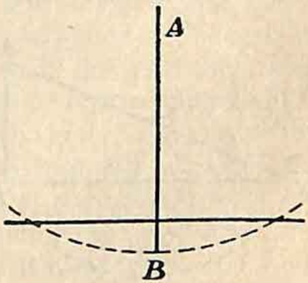


FIG. 5

that the shortest distance from an external point to a line is the perpendicular.

(iv) Isosceles triangle (Fig. 6). One end of a convenient length of tape (or string) is fixed at *A*. The other end swings to cut the base line at $B_1$ and $B_2$. The midpoint, *C*, of $B_1B_2$ is the foot of the required perpendicular. (The line from the apex of an isosceles triangle to the midpoint of the base is perpendicular to the base.)

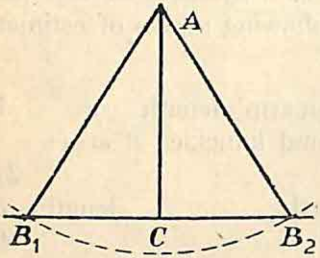


FIG. 6

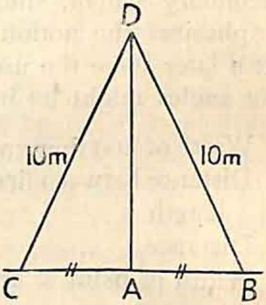


FIG. 7

Another method allows a perpendicular to be erected from a point on the line. Equal distances, *AB*, *AC*, are set out on the base line on either side of *A* (Fig. 7). The two ends of a convenient length of tape (e.g. 20 metres) are held at *B* and *C*. The midpoint, *D*, of the tape is taken out until the tape is stretched tight. *DA* is the required perpendicular.

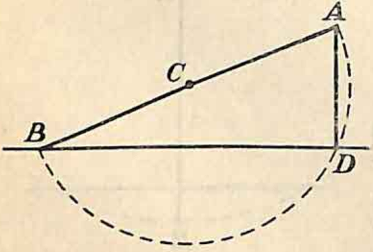


FIG. 8

(v) Angle in a semicircle (Fig. 8). A length of tape is stretched from *A* to a point *B* in the base line. The midpoint, *C*, of *AB* is fixed. The end of the tape, $B$, swings around $C$ to cut the base line again at $D$, the foot of the required perpendicular. $\angle ADB$ is the angle in a semicircle. This method is quick and simple to use.

## EXERCISES IN SURVEYING

Ideally surveying exercises should be planned so that each can be completed in one working period. It is often very difficult to stop work on an exercise and continue it some days later. When exercises will obviously occupy more than one period, some thought should be given to devising convenient methods of stopping and restarting work. Records must be full and accurate so that pupils may work from them some weeks later. The quality of recording is often improved if pupils are made to work from each other's records, i.e. group $A$ draws plans from group $B$'s records, and vice versa. Each piece of recording should carry a title and date, together with a rough plan or diagram illustrating the exercise. The recorders' names should also be given.

Although at times children may work in groups on the same exercise, it is very rarely that a whole class will be working together. At other times the groups can be carrying out different exercises. This avoids the need for a large amount of equipment, since only one or two groups will be using a particular form of instrument at the same time. The size of the groups depends partly on the nature of the exercise. As detailed on page 182 a group of eight pupils is suitable for offset surveys. For height-finding this might be divided into two groups of four each. The composition of a group should remain fairly constant, so that it forms a team working together.

When a large amount of surveying work is done in a school a system of work-cards can be most valuable. Each work-card gives brief details of an exercise to be carried out by one group and also a list of the required apparatus. The work-card is given to the group leader who organizes his group accordingly. This system will be found to save much of the time needed to explain to groups the nature of their particular tasks. One or two years would be needed to build up such a system, but the effort is well repaid. A record can be kept showing which exercises each group has carried out.

## 1. SETTING OUT AND MEASURING LINES

(*a*) *Indirect measurement.* The procedure in simple cases of direct measurement of lines has already been described on page 167. A number of difficulties can arise in practice and should be allowed to do so, for they often illustrate fresh mathematical ideas. Sometimes it is impossible to measure a distance directly by tape owing to the presence of a pond, river, or other obstacle. The following methods can be used:

(i) To find $AB$ (Fig. 9). Set out angles of 60° at $A$ and $B$ as shown. (String equilateral triangles, e.g. 2 m, 2 m, 2 m, will do this.) Find the point of intersection, $P$.

$AB = BP$ (sides of an equilateral triangle).

(ii) Set out an angle of 90° at $A$, and an angle of 45° at $B$ (Fig. 10). Find the point of intersection, $P$.

$AB = AP$ (sides of an isosceles triangle).

(iii) Methods (i) and (ii) are possible only when $A$ is visible from $B$. When $A$ and $B$ are not intervisible the following method can be used (Fig. 11).

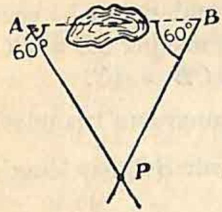


Fig. 9

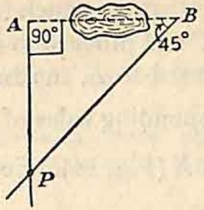


Fig. 10

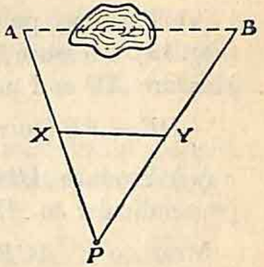


Fig. 11

Take a point, $P$, from which both $A$ and $B$ are visible, and where $PA$ and $PB$ can be measured. Find the midpoints, $X$, $Y$, of $PA$, $PB$.

$AB = 2XY$. (The line joining the midpoints of two sides of a triangle is half the third side.)

$XY$ is also parallel to $AB$, and it gives the direction of $AB$.

(iv) Set out $BD$, any distance, perpendicular to $AB$, and mark its midpoint, $C$, with a ranging rod (Fig. 12). Set out $DE$ perpendicular to $BD$. Move along $DE$ to the point $F$ where, $F$, $C$, $A$ are all in the same line of sight.

$AB = DF$ (corresponding sides of congruent triangles).

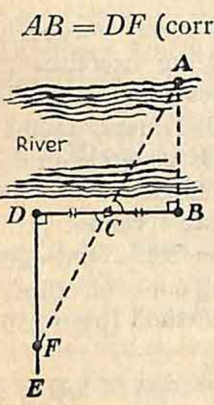


Fig. 12

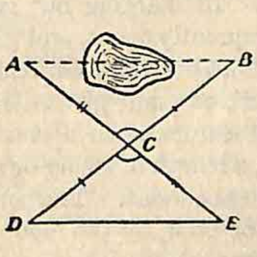


Fig. 13

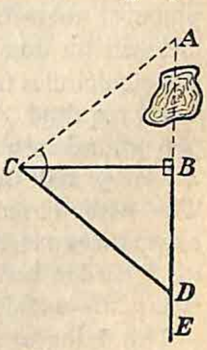


Fig. 14

(v) Take any point $C$ from which $A$ and $B$ can be seen (Fig. 13). Measure $BC$ and proceed to $D$, making $CD = BC$. Measure $AC$ and proceed to $E$, making $CE = AC$.

$DE = AB$ (corresponding sides of congruent triangles).

(vi) Produce $AB$ to $E$ (Fig. 14). Set out $BC$, any length, perpendicular to $AB$.

Measure $\angle ACB$.
Set out $\angle BCD$ equal to $\angle ACB$.

$AB = BD$ (corresponding sides of congruent triangles).

(vii) One common method of measuring distances at sea is by measuring the angle subtended by an object of known height, e.g. a lighthouse. Tables are given in nautical almanacs for converting subtended angles to distances. This method is not of great value on land, but the problem is of interest. A set of tables could be compiled by a class working in groups. Drawing or calculation may be used. For large distances, radian measure can be used. The error incurred by assuming the height to be an arc of a circle is negligible.

(*b*) *Loci*. The setting out of a locus is rarely needed in practical map-making but many interesting exercises in loci can be done. In marking out games pitches, circles and semicircles frequently occur, and parallel lines are sometimes required. The drawing of a really large circle on the playground in chalk or paint gives a satisfaction which the use of pencil and compass on a small scale cannot give. The method used, a length of string or rope fixed at one end, emphasizes a circle as a locus. The setting out of an ellipse, for a garden bed or pool, by the 'loop' method (page 250) is equally satisfying.

The fixing of points through the intersection of loci is a natural development. A point which is 2 m from one wall of the playground and 1·5 m from another wall is found at the intersection of lines parallel to the two walls. The drawing of a plan showing the point and how it was obtained can then follow.

Much of the class-room work on loci would benefit considerably by similar practical exercises taken out of doors first of all.

## 2. AREA SURVEYS

The measuring and drawing to scale of various plots of ground, etc., will form a major part of the surveying work done in schools. The plans should be labelled according to the notes on conventional drawing practice on page 247. If maps of large areas are drawn the symbols used for roads, railway lines, etc., should conform to those used in standard maps.

The following exercises are approximately in order of difficulty.

(*a*) *Measuring and drawing to scale simple rectangular shapes where the corners are assumed to be right angles*, e.g., class-room, school hall, games pitch, etc. The use of squared paper for drawing can ease scale and drawing difficulties for children of low ability.

(*b*) *Marking out a games pitch, vegetable plot, or other rectangular area.* In this case adjacent sides must be set out perpendicular to each other by one of the methods already described. Other shapes can also be set out, triangles, circles, ellipses, etc. The plan of the shape to be set out may be drawn in the class-room before the work is done. It can then be used as a reference when setting out the actual plot. It may also be valuable in helping preliminary discussion. 'Here is the shape we want to mark out. How are we to do it?' This reading of a plan in preliminary discussion is excellent practice. Not only is the plan given concrete meaning, but the real operations needed to set out the plot are also brought to mind. It is a perfect translation of symbolism into actual fact.

(*c*) *Measuring and drawing to scale other shapes with straight sides*, e.g. quadrilateral playground, triangular grass plot, etc. If length measurement only is used, quadrilaterals and polygons must be subdivided into triangles to fix their shapes. Drawing the shape involves the drawing of a triangle given the three sides. A beam compass is useful for drawing arcs of large radius.

If angle measure is also used the following methods are possible.

(i) From an approximate central position in the area, the bearings and distances of the corners are measured (Fig. 15). If an anglemeter is used, it can be set with zero line pointing to *A*. Readings are then taken for the other three corners. In this case we are fixing the positions of the four corners by their bearings from *O* and their distances from *O*. The drawing involves the construction of a triangle from two sides and the included angle.

The directions might be measured as compass bearings taken with a sighting compass.

This is one of the easiest methods, and is capable of a reasonable degree of accuracy.

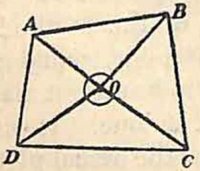


FIG. 15

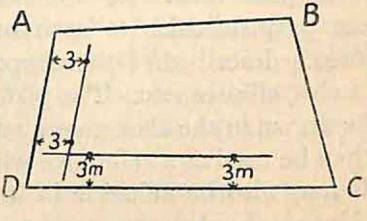


FIG. 16

(ii) The lengths of the four sides can be measured, together with one angle. The measurement of a second angle can act as a check. If the sides of the area are walls or fences, the measurement of one of the angles is not easy to carry out practically. Rods can be placed inside the area at equal distances from the walls (Fig. 16). These will give lines parallel to the walls and the angle between the lines is easily measured. This exercise is valuable as a forerunner to the use of survey lines.

(iii) A measured base line, $XY$, is marked out somewhere in the middle of the area (Fig. 17). Using an angle-measuring instrument the directions of $A$ from $X$ and from $Y$ are found. The directions may be measured as compass bearings or by measuring $\angle AXY$ and $\angle AYX$. This is repeated for each corner. The drawing involves the construction of a triangle ($AXY$) from one side and two angles. Great care is needed in this method; small errors in measuring the angles can cause large errors in the plan. The line $XY$ should not point towards any of the corners, or other features to be plotted.

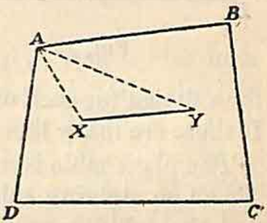


FIG. 17

(iv) The method of the plane table is similar to (iii) but the drawing is constructed on the spot and no readings are taken. A plane table can be made from a large drawing-board adapted so that it can be mounted on a tripod. The base line is marked out in the field as in (iii) and the plane table is set up at one end of the base line, $X$ (Fig. 18). A sheet of drawing-paper upon which the plan is to be drawn is pinned to the plane table. A point is marked on the paper to represent the position of the plane table in the field, and a pin is inserted through the point. A sighting rule is placed against the pin and rotated so that it sights the other end of the base line. A line is then drawn and marked off according to scale, to represent the base line. Sighting lines are then drawn for each of the corners or objects to be plotted. If there are many lines, each one should be labelled.

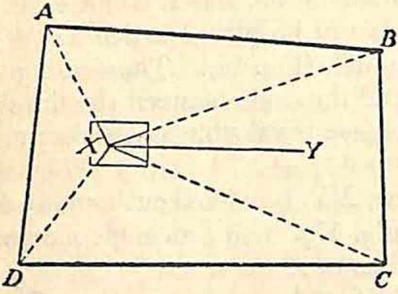


FIG. 18

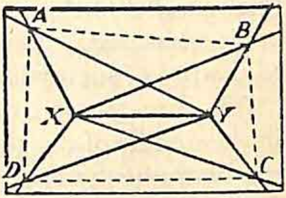


FIG. 19

The plane table is moved to the other end of the base line, *Y*. The sighting rule is placed along the drawn base line and the table rotated until the directions of the real and drawn base lines coincide. Sighting lines are then drawn for each corner or object from *Y* (Fig. 19). The intersection of a pair of sighting lines fixes the position of each object. Details can be filled in by measuring with a tape.

This method, like (iii), requires great care if it is to be of any value. A plane-table method may also be used in (i) by drawing sighting lines from a central point instead of measuring angles. The distances are still measured as in (i) and marked off along the sighting lines.

(*d*) *Offset surveys* can be used when it is inconvenient to measure the actual boundaries of an area or where the boundaries are not straight.

Ranging rods are placed at short convenient distances from the fences to form the corners of a framework of survey lines (Fig. 20). Measurements of the survey lines are made.

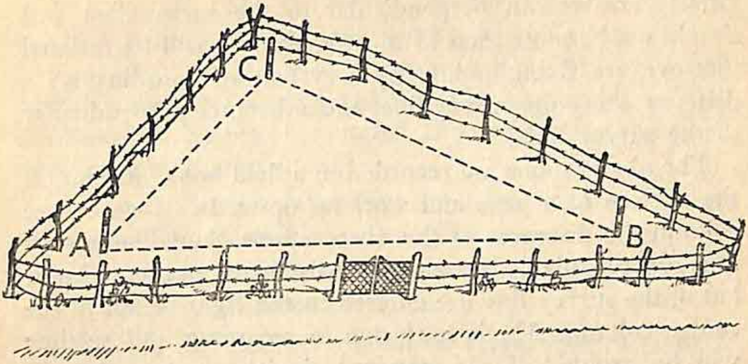


FIG. 20

The distance of the fence from the survey line is also measured wherever it is necessary, together with the distance along the survey line of each observation. Where there are straight fences, two or three of these offsets would be measured (Fig. 21). In other cases the number of offsets depends upon the shape of the boundary fence (Fig. 22).

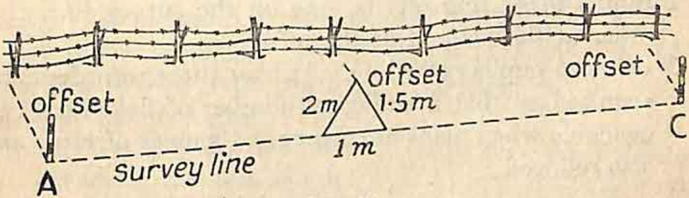


FIG. 21

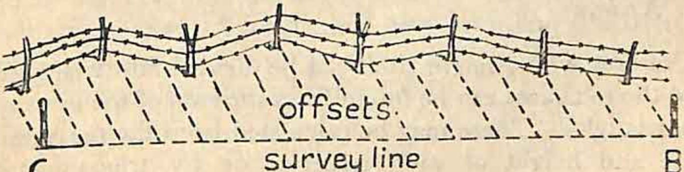


FIG. 22

Offsets are set out perpendicular to the survey line and should not be more than 15 m in length. It will be realized that we are fixing points by a system of coordinates: a distance along the survey line, and a distance perpendicular to the survey line.

The observations are recorded in a field book, starting at the bottom of a page and working upwards. In a centre column the distances of the observations along the survey line are recorded. Details and distances to the right or left of the survey line are entered to the right or left of the centre column. Much confusion in recording and reading will be avoided if the standard field-book procedure is followed.

Offsets can first be used to plot details along one side of a playground, or along a path, before going on to areas of three or more sides.

A surveying group can consist of:

1 chief surveyor, who organizes and controls the work,
2 pupils in charge of the survey line tape,
2 pupils measuring offsets, one on the survey line, the other at the far end of the offset,
3 or more pupils recording. At least three recorders are required so that a sufficient number of field books is available when plans are drawn. Chances of error are also reduced.

## 3. AREAS

An area with straight sides can be divided into triangles, and the total area can be found from the sum of the areas of the triangles. These may be calculated from the measured base and height of each triangle, or by trigonometry (page 134).

An area can also be found by drawing the plan on squared paper to some convenient scale. If the scale is 1 cm = 10 m, the $\frac{1}{10}$-cm or 1 mm square on common graph paper represents 1 square metre. The number of squares contained in the plan is counted, following some organized system. For example:

(*a*) Count the number of whole 1-cm squares. (Each represents 100 sq. m.)

(*b*) Count the number of whole 1-cm columns of 1 mm width. (Each represents 10 sq. m.)

(*c*) Count the number of whole small squares. (Each represents 1 sq. m.)

(*d*) Count the number of part small squares, and divide by 2. (The average size of these part squares will probably be ½ small square.)

Another method is to divide the plan into a series of parallel strips of equal width (Fig. 23). A centre line is drawn in each strip and measured. The area of each strip is found by multiplying its width by the length of the centre line. The total area is then found from the sum of the strips. The calculation may be carried out in other ways; the total length of the centre lines can be multiplied by the strip width, or the average length of the centre lines multiplied by the total width of the plan. This method provides a very satisfactory approximation for the area. Its accuracy is increased by increasing the number of strips used.

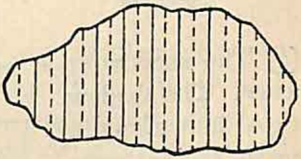


FIG. 23

## 4. MEASURING HEIGHTS

Apart from showing geometrical ideas, the measurement of heights presents an excellent introduction to trigonometry (page 129). The following methods of measuring heights can be used.

(*a*) A base line is measured from the foot of the object whose height is required (Fig. 24). The angle of elevation of the top of the object is measured with clinometer, astrolabe, etc. The height of the object is then found by scale drawing or by trigonometry.

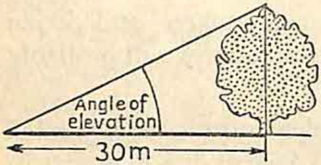


FIG. 24

It is generally necessary to add the height of the observer's eye-level to the height as found from drawing or calculation. This also applies to most of the other methods. The drawing involves the construction of a triangle from one side and two angles.

(*b*) The shadow stick method (Fig. 25) is of historical interest. The shadow stick, of known length, is set up erect at some convenient point. The length of its shadow is measured, also the shadow of the object whose height it is desired to find. For the sake of clarity, the ratio of the length of the stick to the length of its shadow should

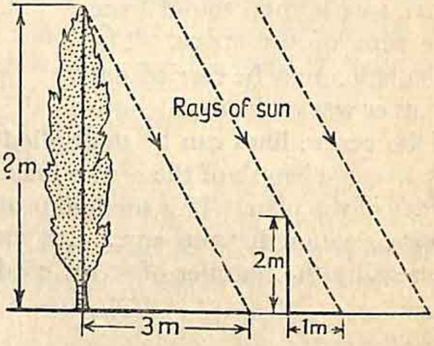


FIG. 25

be a simple one, e.g. 2 : 1, 1 : 3, etc. The calculation uses ideas of similarity.

(*c*) The ' boy scout ' method (Fig. 26) also uses similarity. A rod is placed at *B*, 9 paces from the foot of the object. The observer lies at *C*, one pace away from the rod. He sights the top of the tree, and the point, *D*, where the line of sight cuts the rod is noted. *BD* is measured in centimetres. The number of centimetres gives the height of the tree in decimetres, which can then be converted into metres.

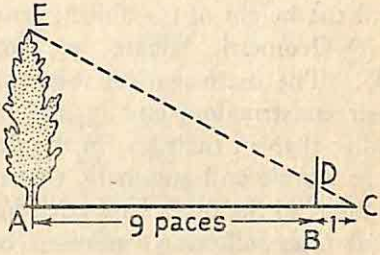


FIG. 26

$$\frac{AE}{BD}=\frac{AC}{BC}=\frac{10}{1}.$$

$\therefore AE = 10BD$. (Each centimetre in *BD* represents a decimetre in *AE*.)

(*d*) The set-square method is a very simple application of similarity to the problem of measuring heights (Fig. 27). The observer moves out from the base of the object until the top of the object can be sighted along the sloping edge of a set-square. The set-square must be held so that the sides which contain the right angle are horizontal and vertical.

The easiest form of set-square for use is the isosceles set-square with its vertical and horizontal sides equal. The height of the object is then given by the observer's distance from the object (plus height of observer's eye-level).

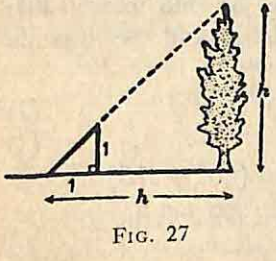


FIG. 27

Other set-squares may have their vertical and horizontal sides in the ratio $1:\sqrt{3}$, $1:2$, $1:3$, etc., and these generally give more accurate results than the isosceles set-square. As the observer's distance is divided by $\sqrt{3}$, 2, 3, etc., to find the height of the object, errors will also be divided.

(*e*) Geometric square: see notes on instruments, page 195. The mathematical value of these instruments lies in their construction, and in understanding how they work, rather than in their use in the field. In practice their use is so simple and automatic that no mathematical ideas are brought to the observer's attention.

(*f*) The following variations of the above methods are employed in the more difficult situation when the base of the object is inaccessible and the distance from its base cannot be measured.

(i) Two readings of the elevation of the top of the object are measured by clinometer, at *A* and at *B* (Fig. 28).

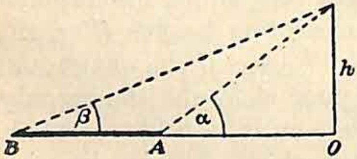


FIG. 28

*O*, *A*, and *B* must lie in the same straight line, i.e. *B* is placed on the line of sight *OA*. *AB* is measured and a scale drawing made to find the required height. The solution by trigonometry is given below. Some of the more able pupils should be capable of using the formula obtained.

$$h = OA \tan \alpha \quad . \quad . \quad . \quad . \quad (1)$$
$$h = OB \tan \beta \quad . \quad . \quad . \quad . \quad (2)$$
$$\therefore OA \tan \alpha = OB \tan \beta = (OA + AB) \tan \beta$$
$$= OA \tan \beta + AB \tan \beta$$
$$\therefore OA \tan \alpha - OA \tan \beta = AB \tan \beta$$

$$OA(\tan\alpha - \tan\beta) = AB\tan\beta$$

$$\therefore OA = \frac{AB\tan\beta}{\tan\alpha - \tan\beta}$$

Substitute for $OA$ in (1):

$$h = \frac{AB\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$$

$$= \frac{AB\sin\alpha\sin\beta}{\sin(\alpha - \beta)}$$

which is a form more suitable for logarithmic computation.

(ii) Two readings are taken along the same straight line from the base using the 1:1 and 1:2 set-squares (Fig. 29).

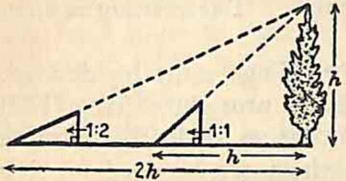


FIG. 29

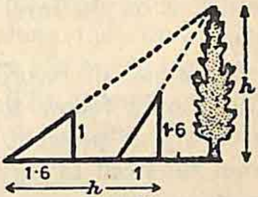


FIG. 30

The distances between the two points of observation is equal to the height of the object.

(iii) A single set-square with sides in the ratio 1·6 : 1 (angle 58¼°) may be used instead of the two set-squares of (ii). Two readings are taken using the set-square both ways (Fig. 30). The distance between the two points of observation is equal to the height of the object.

## 5. LEVELLING

Simple surveys take no account of the various heights of the land which is being mapped. The process of finding or comparing heights is known as *levelling*. The levelling instruments described on page 198 fall into two classes; those which give a horizontal line of sight (U-tube level, horizontal sighting rule) and those which test the level of a rod or string (Egyptian level, spirit-level). When an instrument of the first type is used it may be set up at one end of the line to be levelled; a levelling staff is placed at suitable points along the line and readings are taken. These will include heights and distances along the survey line.

It will generally be found necessary for the pupil holding the levelling staff to take the height reading, since the staff will be too far from the sighting instrument for the calibrations to be read. A simple method is for the pupil holding the staff to move a card up or down the staff as directed, until it is on the level line of sight. The reading is then taken.

A method of recording the readings must be devised. This should follow the field-book procedure (page 182) as closely as possible. Easy instances of levelling which avoid the need to survey and which can be used for first practices are:

(*a*) Finding the levels along a straight sloping path or road.
(*b*) ,, ,, ,, ,, line across a field or playground (i.e. making a section). This is made more interesting by the presence of a mound or hollow. Even in these cases the method of recording should follow field-book practice so that new methods do not have to be described later.

It is not often that a levelling exercise will start from a point whose height above sea-level is known. The starting-point is therefore generally taken as zero level and the other points compared to it. In the recording shown the line of sight gives a reading of 92·5 cm on the levelling staff at the beginning of the survey line. At a distance of 3 m the reading

| *Distance along S.L.* | *Staff reading* | *Level* |
|---|---|---|
| 16·8 m | 122·5 cm | −30 cm |
| 12·6 m | 100 cm | − 7·5 cm |
| 8·1 m | 75 cm | +17·5 cm |
| 5·4 m | 55 cm | +37·5 cm |
| 3·0 m | 82·5 cm | +10 cm |
| 0 | 92·5 cm | 0 |

was 82·5 cm, showing that the ground had risen 10 cm. At 5·4 m the staff reading is 55 cm, showing a rise of 37·5 cm from the starting-point. The three columns take the place of the centre column of the field book; survey details are entered on either side as in normal field-book procedure.

When the distance from the level becomes too great for accurate reading (generally about 15 m with plain sights), the level must be taken forward along the survey line. This is also done on steep slopes when the line of sight falls below or above the levelling staff. The record shows the method used. At a distance of 14·4 m along the survey line the reading on the levelling staff is 12·5 cm and the ground level is 62·5 cm above the starting-point. The staff is kept at the 14·4 m position while the level is brought forward. Another reading, 90 cm, is then taken from the new position of the level, and this is used as the start of the next set of observations. At 18 m the staff reading is 75 cm, so the level at 18 m will be (62·5 + 15) = 77·5 cm above the starting-point of the survey line.

| *Distance* | *Staff* | *Level* |
|---|---|---|
| 18 m | 75 cm | 77·5 cm |
| 14·4 m | 90 cm | 62·5 cm |
| 14·4 m | 12·5 cm | 62·5 cm |
| 10·5 m | 25 cm | 50 cm |

The level does not have to be placed on the survey line. Its horizontal line of sight is independent of position, and when a curved path such as *ABC* (Fig. 31), is to be levelled, an obvious economy results from placing the level at a position such as *O*.

When the Egyptian or spirit-levels are used, the readings of heights are taken at regular intervals along the survey line, corresponding to the length of the rod or string between the two levelling staffs. A form of recording for this method is shown. As before, two columns showing heights are placed on one side of the distance column of the field book. The rise or fall between each pair of points is recorded in the column next to the distance column. The total rise or fall is given in the right-hand column. The columns should not be too wide, so that ample space is left on the page for entering survey details to the right and left.

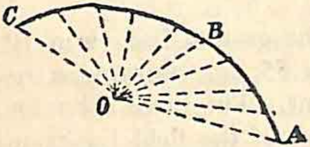


FIG. 31

| *Distance* | *Rise* | *Level* |
|---|---|---|
| 12 m | 7·5 cm | +57·5 cm |
| 9 m | 20 cm | +50 cm |
| 6 m | 17·5 cm | +30 cm |
| 3 m | 12·5 cm | +12·5 cm |
| 0 | .. | 0 |

## SURVEYING INSTRUMENTS AND APPARATUS

The instruments described here are of a simple nature, and many of them can be made without much difficulty. The more complicated instruments (theodolite, sextant) should only be used in the third and fourth years of a four-year surveying course. In all other cases they should be pvoided since their construction often obscures the basic arinciples we wish the pupils to understand.

### MEASUREMENT OF DISTANCE

(*a*) *Chain.* A surveyor's chain, 66 feet or 20·1168 metres long, is divided into 100 links. A brass tablet marks every tenth link, the shape of the tablet showing the distance from the nearer end of the chain (Fig. 32). A chain should be dried after use before it is put away. When folding a chain for storage, first lay out the chain in a straight line and then bring the two ends together so that the chain is doubled. Next, starting at the middle of the chain, fold so that the pairs of links lie side by side. A strap or band should be tied around the finished bundle (Fig. 33).

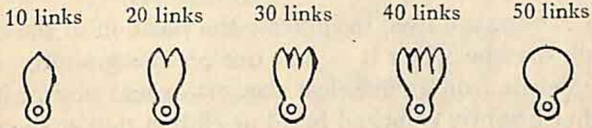


FIG. 32

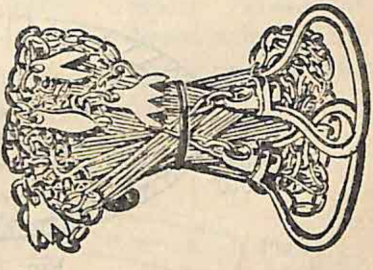


FIG. 33. Surveyor's Chain

(*b*) *Tapes.* It is better to have one or two tapes of good quality and to supplement them with lengths of string or wire rather than to buy many tapes of inferior quality. The latter rarely bear hard use, and often become inaccurate if they get wet. All tapes should be left to dry before storing. Care should also be taken that they do not become folded or creased when they are rolled. This can be prevented by passing the tape between two fingers as it is drawn in.

(*c*) *Ranging rods* are used to mark the ends of survey lines, etc. They are also used for measuring short distances. They should be about 2 m in length, divided into 0·5 m or 1-link sections, and preferably brightly painted so that they can be easily seen. One end should be sharply pointed so that the rod can be placed in the ground. An iron point is very useful, but not easy to make.

(*d*) *Arrows* are used to indicate the position of the end of a chain or tape when it is laid out on the ground. They can be made from metal meat skewers or tent pegs, with red or other brightly coloured braid or ribbon tied at the top so that they may be easily seen.

## MEASUREMENT OF ANGLES

(*a*) *Anglemeter* (Fig. 34). The scale can be copied from a protractor, or cut from a sheet of polar graph paper.

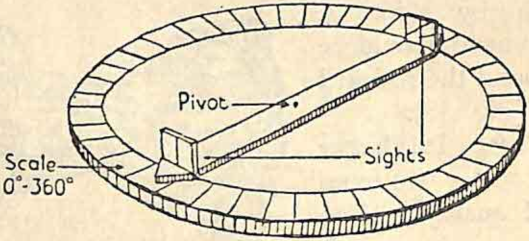


FIG. 34

The latter is obtainable from firms selling drawing-office materials. The sights can be made of perspex blocks with a vertical line scratched on each, or metal frameworks can be constructed having a vertical centre wire or slit; or, very simply, thin nails will suffice. Care must be taken to ensure that the centre of the scale and the pivot of the sighting arm coincide. The scale should be mounted first, and its centre found by drawing a number of diameters. The pivot must then be carefully placed at this point. Accuracy can be tested by taking readings at both ends of the sighting arm. They should always differ by 180°.

(*b*) *Astrolabe* (Fig. 35). This is a very old instrument, used in astronomy and navigation. Its construction is very similar to that of the anglemeter. The astrolabe, however, is suspended from a ring and therefore forms its own plumb-line. Care must be taken that it hangs correctly. Its shape must be a true circle, and the sighting arm must be evenly balanced about its centre point.

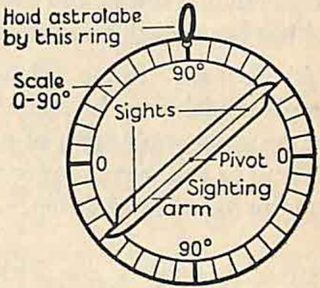


FIG. 35

(*c*) *Bearing compass, or sighting compass.* These give readings of magnetic bearings, i.e. a direction is stated in terms of the clockwise angle from magnetic north. They

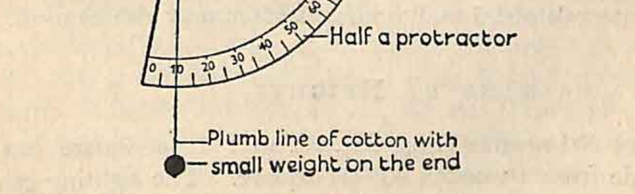

Pins or small nails to act as sights
Piece of wood
0
10
20
30
40
50
60
70
80
Half a protractor
Plumb line of cotton with small weight on the end

FIG. 36

can sometimes be purchased cheaply from stores dealing in ex-government surplus material. They must not be used near any iron object.

(*d*) *Clinometer* (Fig. 36). This is one of the easiest instruments to make. The plumb-line of cotton or fine string must hang from the centre of the protractor scale. The weight should not be too small, nor the line too long, or the lightest breeze will make it difficult to take readings.

(*e*) *Sextant.* Some schools may possess a sextant, or simple models may be made in conjunction with the science course as an application of mirror reflection. They are not difficult to use, but the way in which they measure angles is obscure for many children.

## CONSTRUCTION OF PERPENDICULARS

(*a*) *Surveyor's cross* (Fig. 37). This is a wood cross holding two pairs of sights arranged so that their lines of sight are perpendicular to each other.

(*b*) *Right-angled triangle.* A 3·6 m loop of string is knotted to give lengths of 0·9 m, 1·2 m, and 1·5 m. When the string is stretched to form a triangle with the knots as its vertices, the angle between the 0·9 m and 1·2 m sides is a right angle. A 9 m loop of string with knots at intervals of 1·5 m, 3·6 m, and 3·9 m may also be used.

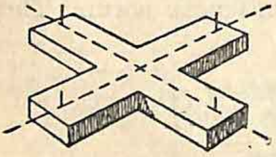


FIG. 37

## MEASUREMENT OF HEIGHTS

(*a*) *Set-squares* (Fig. 38). The 1:1 set-square can be made from a wooden 45° set-square. The sighting can be done along the sloping edge, but this is not easy. A pair of pins or thin nails may be mounted as sights on a line parallel to the sloping edge. A very simple alternative is a milk straw held on the sloping edge by two strips of cellotape, and used as a sighting tube.

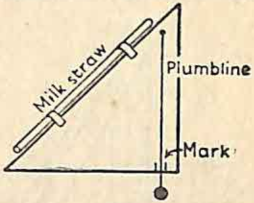


FIG. 38

The correct position of the set-square can be tested by a plumb-line, or the set-square can form its own plumb-line. In the latter case the centre of gravity of the set-square is found by drawing the medians (Fig. 39). A hole is bored directly above the centre of gravity, which lies at the inter-section of the medians. The instrument is held on a pin through the hole and hangs in the correct position. If sights are mounted on this form of set-square, they must not interfere with its balance. The construction can lead to a useful discussion on centres of gravity; the example is a very simple one.

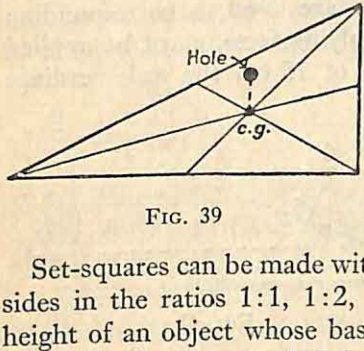


FIG. 39

Set-squares can be made with their vertical and horizontal sides in the ratios 1:1, 1:2, 1:3, etc. For measuring the height of an object whose base is inaccessible, a set-square with sides in the ratio 1:1·6 can be used (page 187).

(*b*) *Geometric square* (Fig. 40). This consists of two perpendicular arms, each about 30-32 cm long. A scale 30 cm long is set out on one arm, and is divided into units of 1 cm. It is labelled 0, 1, 2, . . ., 30 cm. Two sights are placed one at each end of the scale. On the other arm a plumb-line is suspended from a point 30 cm above one end of the scale.

A distance of 30 cm is measured from the base of the object whose height it is desired to find. The top of the object is sighted along the pair of sights on the bottom arm. The position of the plumb-line on the scale gives the height of the object in centimetres (observer's eye-level to be added).

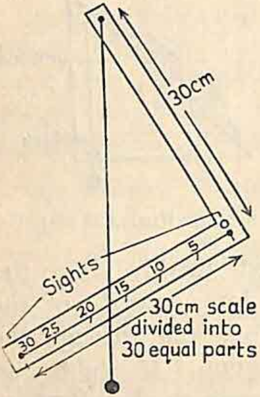


FIG. 40

It will be noticed that as the plumb-line swings across the scale, it does not remain perpendicular to the scale. The calibration marks should, therefore, not be perpendicular to the scale, but follow the direction of the plumb-line.

If base-lines other than 30 cm in length are used, a corresponding multiplying factor must be applied to the scale. For a base-line of 15 cm the scale readings are halved.

The principle underlying this instrument is as follows:

In Fig. 41 the scale reading on the geometric square is *EF*; *DF* = 30 cm.

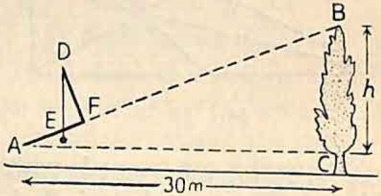


FIG. 41

The triangles *ABC*, *DEF* are similar ($\angle F = \angle C$, $\angle E = \angle B$)

$$\frac{EF}{BC} = \frac{DF}{AC}$$

$$\frac{30 \text{ cm}}{30 \text{ m}} = \frac{1 \text{ cm}}{1 \text{ m}}$$

i.e., each cm along *EF* corresponds to 1 m on *BC*.

The geometric square described above, although simple in construction and use, presents the difficulty that the triangle formed on the instrument (*DEF*) is not in the same position as the triangle in space (*ABC*). This can be overcome by the form of construction shown in Fig. 42, which allows the basic principle to be seen far more readily. Some method of ensuring that the instrument is held correctly must be employed: either a plumb-line on the vertical arm or a spirit-level set in the horizontal arm.

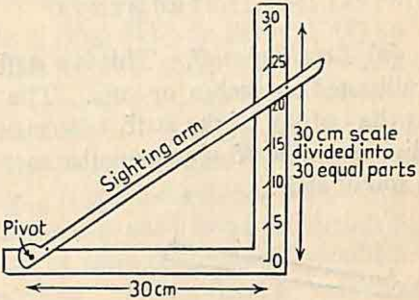


FIG. 42

## AREA SURVEYS

(*a*) *Plane table*. This generally consists of a large drawing-board which can be mounted on a tripod. When plane tabling is to be done in the immediate vicinity of the school, an ordinary table might be substituted. It will provide a larger surface, and a greater measure of stability.

(*b*) *Sighting rule or alidade* (Fig. 43). A sighting rule can be made from an ordinary ruler with a pair of sights mounted on it.

(*c*) *Beam compass* (Fig. 44). When drawing maps or plans, it is often necessary to describe arcs of a radius beyond the range of the ordinary school compasses. A beam compass is easily made from a strip of wood about 50-75 cms long. A point, taken from a pair of compasses, is fixed at one end. A pencil is placed in some form of clip which can be moved along the beam.

## LEVELLING INSTRUMENTS

(*a*) *Levelling staff*. This is a staff, 2 m or more in length, calibrated in inches or cms. The zero reading should be at the bottom of the staff. Because of the large number of divisions, a staff of rectangular section is to be preferred to a round staff.

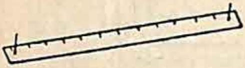


FIG. 43

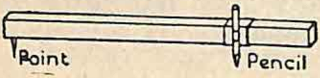


FIG. 44

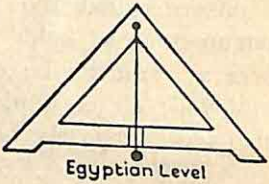


FIG. 45

(*b*) *Egyptian level* (Fig. 45). When the feet of the level are placed on a horizontal surface, the plumb-line coincides with a mark at the centre of the crossbar.

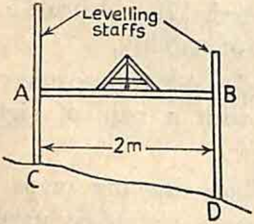


FIG. 46

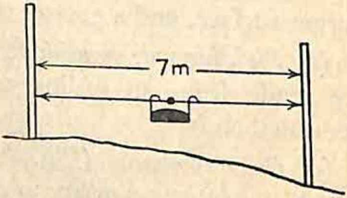


FIG. 47

In levelling, the Egyptian level is used to adjust a rod $AB$, 2 m or 3 m long, to a horizontal position (Fig. 46). The difference in the readings of the levelling staffs at $A$ and $B$ gives the change in ground level from $C$ to $D$.

(*c*) *Spirit-level.* An ordinary spirit-level may be used in exactly the same way as the Egyptian level.

(*d*) *String and bubble level* (Fig. 47). In methods (*b*) and (*c*) heights are found at intervals of about 2 m depending on the length of the rod. This interval may be too short for convenience on long survey lines; a 6 m length of string might be substituted for the horizontal rod. A knot is tied at the centre of the string. A spirit-level in a simple wire cradle is hung at this point, and is used to adjust the string to a horizontal position. The level in its cradle should be tested for accuracy before using. This is done by reversing it on the string; it should give the same reading in both positions.

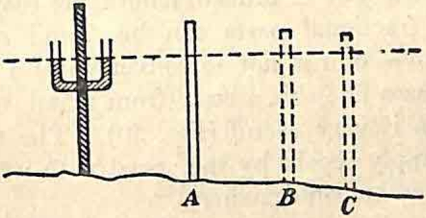


FIG. 48

(*e*) *U-tube level* (Fig. 48). The levels of coloured water in a U-tube give a horizontal line of sight. For convenience the U-tube is mounted on a rod. The readings of a levelling staff are taken at $A$, $B$, $C$, etc.

(*f*) *Horizontal sighting rule* (Fig. 49). This is particularly useful. A pair of horizontal sights are mounted at the ends of a strip of wood. The rule is placed on some convenient surface, and the adjusting screw turned until the spirit-level shows that a correct position has been attained. The adjusting screw, of course, enables the instrument to be used when mounted on an uneven surface.

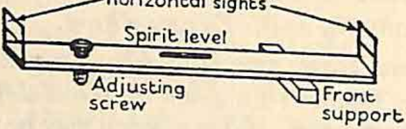


FIG. 49

(*g*) *The theodolite* may also be used for levelling. A horizontal line of sight is obtained when the vertical scale shows a zero reading.

## DRAWING SCALES

The scales used for drawing maps and plans should be convenient to use. Scales of 1 cm = 20 cm, 50 cm, 1 m, 10 m, etc. (or any other units of length) are easy to use since the needed fractional parts can be found on rulers. A scale of 1 cm = 6 m is not so convenient if a ruler is to be used. In cases like this a scale from which measurements can be taken is very useful (Fig. 50). The time taken to make it is amply repaid by the speed with which distances can be marked out on drawings.

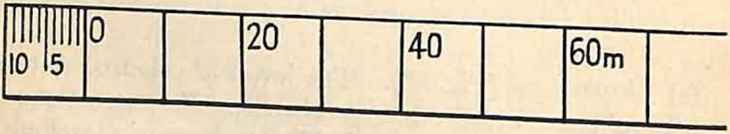


FIG. 50

The first step in making such a scale is to find the length of the 10 m divisions. If a scale 1 cm = 6 m is taken, the unit to represent 10 m can be found as follows:

1 cm = 6 m is the same as 2·5 cm = 15 m. Hence a distance of 2·5 cm can be divided into 15 divisions, each

division to represent a meter. A length equal to 10 such divisions, will be the unit in the scale to represent 10 metres.

This can be done by the standard geometrical construction to divide a line into any number of equal parts (Fig. 51).

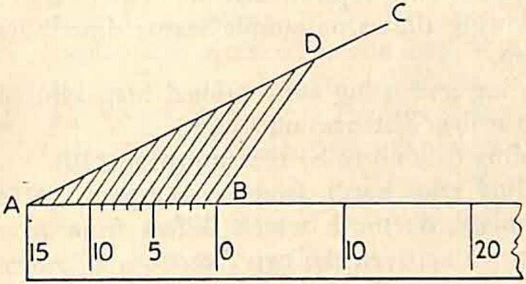


FIG. 51

A line *AC* is drawn at any angle to the scale *AB*. With dividers, or compasses, fifteen equal divisions are marked off along *AC*; these divisions can be of any length. The fifteenth division mark, *D*, is joined to *B*. Lines parallel to *DB* are drawn through each of the other divisions along *AD*. These parallels will divide *AB* into fifteen equal parts.

Distances are taken off the scale with dividers or compasses. If we wish to mark off 43 m, one point of the dividers is placed on the 40 m mark and the dividers opened until the other point reaches the 3 m mark in the divided end section (Fig. 52). The dividers are then opened to 43 m according to the scale.

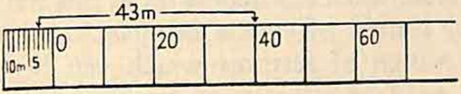


FIG. 52

## MAP WORK

Map work can be closely linked with practical surveying, much of which is concerned with map-making. The following list suggests some typical exercises in general map work.

(*a*) Following routes on simple maps; describing routes from a map.

(*b*) Reading and using conventional map symbols.

(*c*) Map scales; distances on a map.

(*d*) Finding true north by the sun (page 210).

(*e*) Finding true north from a compass. A compass points to magnetic north which differs from true north. The difference between the two directions is known as the magnetic variation. Its value is not constant, but changes from place to place. There is also a slight annual change. The amount of the variation is stated on most maps, also the annual rate of change. The variation given on a map will be its value in the year that the map was printed. A correction must therefore be applied to find the present variation. From this and the direction of magnetic north, the direction of true north may be found.

(*f*) Setting a map with a compass. A map can be used more effectively in open country if it is correctly set. A small compass is placed on the magnetic north line shown on the map. The map is then rotated until the direction of the compass pointer coincides with that of the magnetic north line. All directions on the map then correspond with those in the country that it represents. Features are more readily identified when the map is set in this way.

(*g*) Setting a map without a compass. A map may be set by observation of features which can be recognized. If a straight road or railway can be seen and can also be identified on the map then the map may be rotated until the road or railway on the map lies parallel to the real road or railway.

Another method is to identify two features on the map and in the country. The map is rotated until the features on the map occupy positions corresponding to the real features.

If the observer's position on the map is known, a direction line to any recognized feature can be used to set the map. A ruler or straight-edge is placed on the map so that it connects the feature with the observer's position. The map is rotated until the ruler points in the direction of the real feature. The direction of a second feature can act as a check.

(*h*) Fixing observer's position on a map from nearby features. Two or more features are identified and their positions found on the map. The map is then set and sighting lines are taken connecting the features with their map positions (Fig. 53). The intersection of these lines fixes the observer's position on the map. It is advisable to take three bearings. The lines drawn will not generally pass through a point, but will form a small triangle.

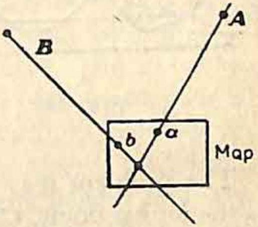


FIG. 53

A better method is to measure the bearings of two or more identified features. Back bearing lines are drawn from these features on the map. Their intersection fixes the observer's position.

(*i*) Contours, slope from contours; profiles and sections.

(*j*) Determining visibility between two points (Fig. 54). The simplest method of determining whether two points are intervisible is to draw a section of the country between them. A line, *AB*, is drawn on the map, joining the two points in question. A section along *AB* is then drawn, and the positions of the two points on the section joined by a straight line. If this line lies clear of the section, the two points are intervisible. In the example shown in Fig. 54 it will be seen that *A* and *B* are hidden from each other by the crest, *C*.

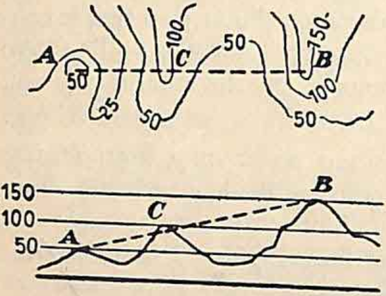


FIG. 54

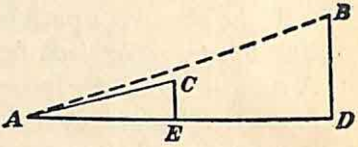


FIG. 55

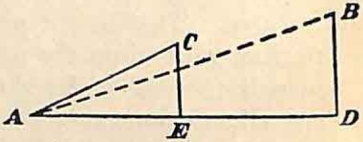


FIG. 56

The problem may also be solved by calculation. If *A* is the lowest point, *CE* (Fig. 55) represents the height of *C* above *A*, and *BD* represents the height of *B* above *A*. If the ratio *BD*/*AD* is greater than *CE*/*AE*, *A* and *B* are intervisible. If *BD*/*AD* is less than *CE*/*AE*, *A* and *B* are hidden from each other (Fig. 56). The ratios *BD*/*AD*, *CE*/*AE*, are the gradients between *A* and *B*, and between *A* and *C*.

$$\left(\text{Gradient} = \frac{\text{Vertical Rise}}{\text{Horizontal Distance}}\right).$$

With a little practice in observing the positions of contour lines, the nature of slopes can be determined (Fig. 57). If the contour lines are closer together on the high ground the slope is concave. Points on it can be seen from one another.

If the contour lines are closer together on the low ground, the slope is convex. Points on it are hidden from each other.

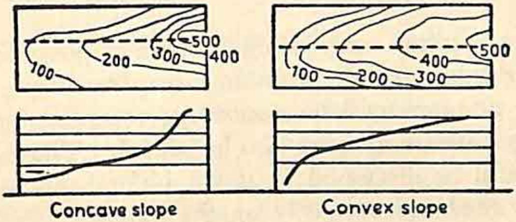


FIG. 57

(*k*) Making contour models. The making of contour models improves the reading of contour maps appreciably. The contours used may be taken from a map, or may be the results of a levelling exercise. One of the simplest methods is the layer method, in which the various heights are shown by layers of wood or thick card. Each contour shape is cut out of card. If the position of the next higher contour line is also marked on it, the fixing of the next layer is made easier. The contour model of Fig. 58 shows land heights in a series of terraces. If desired, the terraces may be filled in with plaster of Paris to give a smooth and more realistic finish.

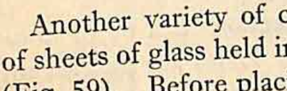


FIG. 58

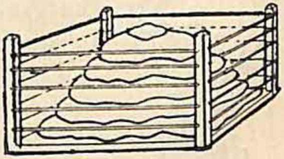


FIG. 59

Another variety of contour model consists of a number of sheets of glass held in horizontal layers in a wooden frame (Fig. 59). Before placing the glass in the frame, each sheet is placed over a map and one contour traced on it. The sheets are placed in the frame in order of height. When viewed from above it presents an excellent three-dimensional effect.

(*l*) Map-reading excursions. When possible, pupils should be taken on an occasional map-reading excursion. A supply of maps will be necessary; compasses and some surveying instruments may also be useful. The work to be done should be discussed in detail beforehand so that no time is wasted unnecessarily on the journey. A complete record of the journey can be made, including times, distances (from map), directions, etc. Some weather details might be observed, wind direction and strength, cloud cover, etc. Exercises can be carried out as opportunities arise on the route.

(*m*) Map projections. It is not possible to give any mathematical treatment of map projections in the high school. It is possible, however, to demonstrate the problem of the geographer who wishes to portray a spherical surface on a plane surface. This may be linked with developments (page 245). Not all solids have developable surfaces; the sphere is an example of a solid whose surface cannot be developed. Some idea of the general methods of solution employed by the geographer may also be given.

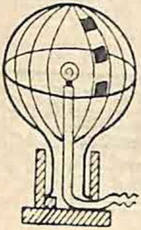


FIG. 60

The impossibility of converting a round surface to a flat one can be shown by cutting up a ball and attempting to lay the sections flat. It follows that all plane maps of the surface of the earth must distort in some way or another.

The three major types of map projections can be demonstrated by a simple model (Fig. 60). Lines are painted on the body of a spherical glass flask to represent the equator and meridians of longitude. An electric-light bulb is held by stiff wires in the centre of the flask, which is held in a wooden stand. The bulb is lighted, and shadows of the equator and meridians are cast on a sheet of paper held close. The three projections are given by:

(i) A cylinder of paper with diameter equal to that of the flask, and held so that it touches the equator.

(ii) a flat sheet of paper held so that it touches the flask at one point.

(iii) a cone of paper placed over the flask. It will touch along a parallel of latitude.

The distortion of areas can be shown by cutting out three squares of paper of equal size and sticking them on the flask in different positions. One should be placed on the equator, another near the North Pole, and the third about midway between them.

Most projections used for maps are modifications of these three. The common Mercator's projection which has been in use since 1569 is a modification of the cylindrical projection.

## XIII

## SUN AND SHADOWS: ASTRONOMY

THE measurement of shadows can be the start of an excellent and prolonged study. Shadow lengths and directions are measured at intervals throughout a sunny day. The shadow may be that of a gate-post or some similar fixed object. The teacher might prefer to undertake the special construction of a shadow board (Fig. 1). This is a large board, minimum size about 60 cm by 45 cm, with a vertical shadow stick, or style, placed at the midpoint of one of the longer sides. The style should have some definite simple height in order to avoid unnecessary calculation later. 25 cm will be found to give too long a shadow for all but very large boards; 10 cm or even 5 cm is better. A piece of paper is pinned on the board, which, with the help of a spirit level, is sited horizontally and in a sunny position. At intervals of about half an hour the position of the tip of the shadow is recorded by a point or cross on the paper, together with the exact time of recording. It is not essential to take the measurements at regular intervals, but the times of recording should be such that they can be conveniently placed on a graph scale; 2.10 is to be preferred to 2.13.

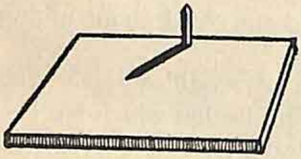


FIG. 1

When the recording is completed for a day, the recorded points may be joined by a line which will give the path of the shadow tip. It is well worth while taking sets of shadow readings at intervals throughout the year. A set of observations about once a fortnight should be aimed at, though, of course, a weekly set gives a higher degree of accuracy.

The following work may be based on the information obtained.

*Stage One.* If the lengths of the shadows are measured they can be plotted against time of day. From this graph can be found the length of the shadow for any given time within the range of observations made. The length of the shadow of any object may then easily be found, e.g. if the style height is 10 cm, shadow lengths at particular times for heights of 10 cm, 20 cm, 2 cm, etc., are found by very simple calculation. This work could be checked by actual measurement.

As both the height of the style and the length of the shadow at a given time are known, the angle of elevation of the sun at that time can be found by drawing (Fig. 2). Angles of elevation at various times of the day and at various times of the year may be compared.

*Stage Two.* A sun-dial of the type used by the Egyptians over 3,000 years ago (Fig. 3) may be constructed from the graph of shadow lengths throughout the day. A vertical style is fixed at one end of a horizontal, rectangular shadow plate. The style may be 2 cm in height, the shadow plate 20 cm long. From the graph already constructed for the 10 cm style the lengths of the shadows of the 2 cm style may be found at the hours and half-hours throughout the day by simple proportion. These distances are marked along the shadow plate and the times placed at appropriate distances. The dial is used by placing it on a horizontal surface with the style towards the sun. It will be realized, of course, that the altitude of the sun varies with the season of the year, and that an Egyptian dial constructed in June is of no value in November (see page 211).

The direction of true North is given by the direction of the shortest shadow. It is not easy to decide directly the position of the shortest shadow. Fig. 4 shows a better method. Using the position of the style on the shadow board (Fig. 1) as centre, two arcs of equal radius are drawn to cut the shadow path. This gives the positions of two shadows of equal length before and after noon. The bisector of the arc which lies between these positions then gives true North.

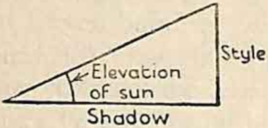
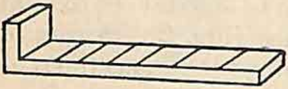


FIG. 2



Egyptian sun-dial

FIG. 3

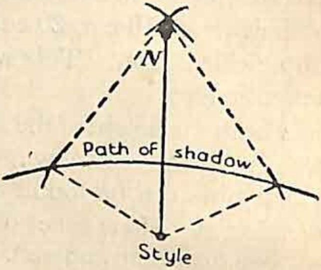


FIG. 4

Earlier we have seen how a drawing method can be used to find the angle of elevation of the sun at a given time of day. This work may now be extended, and by the repetition of many such drawings to find the altitude of the sun at different times in the day, sufficient information may be obtained to draw a graph of time and altitude. This can then be used to give the altitude of the sun at any time of day.

*Stage Three.* The information so far obtained may be combined in a graph of altitude and length of shadow. This is of interest since it can serve as a practical introduction to the ideas of trigonometry. The graph shows the relationship between one angle of a right-angled triangle (altitude) and the adjacent side (shadow length), and its value can be carried far beyond the particular case of shadows of a 10 cm style.

It has already been suggested that readings should be taken throughout the year, but it will probably need a period of two years or more to compile a comprehensive and accurate set of records for the entire year. A graph of noon altitudes of the sun throughout the year is interesting, and provides a way of finding inaccurate readings. The curve should be a smooth one and any readings which cause a deviation from a smooth curve must be considered as errors (see page 91). A graph showing both the noon altitudes of the sun and average temperatures throughout the year is a good introduction to the use of the 'double' graph (Fig. 5.) It is also an excellent method of showing cause and effect.

The graph of the altitude of the sun throughout the year may be repeated for other times of the day, 1 p.m., 2 p.m., etc. This set of graphs is best done by group work, each group contributing the graph of a particular time. The set will provide the basic information needed to construct a set of Egyptian dials, one for each month of the year. The altitudes of the sun for each hour of a day in the middle of each month are found, and shadow lengths of a 2 cm style for each hour are then obtained by drawing. (See also page 217.)

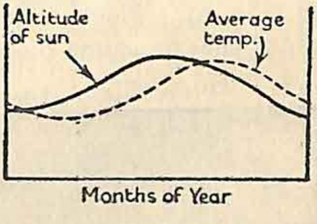


FIG. 5

ASTRONOMY

The work, *Sun and Shadows*, can form the basis of a fuller course in astronomy. A certain amount of the work done under this heading might be termed descriptive mathematics, i.e. we describe or show particular happenings. The way in which day and night or the phases of the moon occur may be shown by a model. The pupils see clearly the reasons for the phenomena, without necessarily carrying out calculations upon them. With able pupils we can introduce an element of quantitative work, e.g., the drawing method of finding the apparent shape of the moon (page 227).

## THE SHAPE OF THE EARTH

Pythagoras (550 B.C.) appears to have been the first person to describe the earth as a sphere. Some three centuries later Eratosthenes was the first to calculate its circumference. It was noticed that at noon on a certain day in the year in Syene, the rays of the sun reached the bottom of a deep well (Fig. 6). The sun was, therefore, directly overhead. Eratosthenes measured the altitude of the sun at Alexandria,

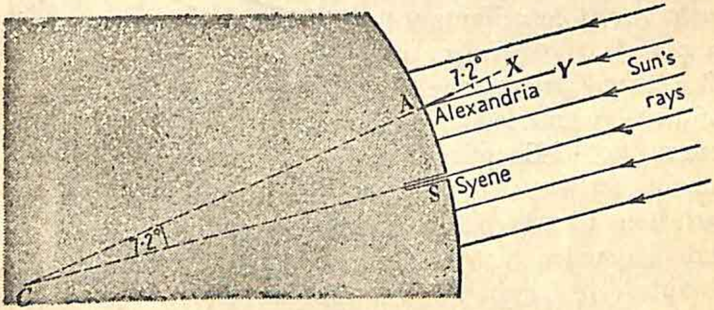


FIG. 6

800 kilometres due north of Syene, at the same time, and found that the rays were inclined at an angle of 7·2° from the vertical.

$\angle ACS = \angle XAY$ (corresponding angles)

$= 7{\cdot}2^\circ$

From Fig. 7:

$$\frac{\text{Circumference of earth}}{800 \text{ km}} = \frac{360^\circ}{7{\cdot}2^\circ}$$

$\therefore$ Circumference of earth

$= 40{,}000$ km.

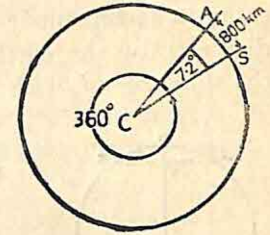


FIG. 7

It is not practicable, of course, to demonstrate the curvature of the earth in a class-room, but many children will have observed a vessel at sea disappearing below the horizon. They may also have noticed that a greater distance can be observed from the top of a cliff than from sea-level.

An important experiment to measure the diameter of the earth is known as the Bedford Level experiment. It was carried out on a 9·66 km stretch of long straight canal in England. Three vertical posts were set up at water-level, one at each end, and the third at the midpoint (Fig. 8). A line of sight cut the two end posts at the same height. A reading taken on the centre post showed a difference of almost 1·83 m from the two end readings.

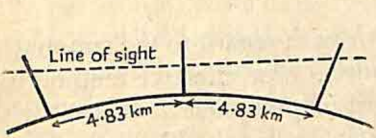


FIG. 8

This information is sufficient to enable us to calculate the diameter of the earth, using the properties of intersecting chords of a circle. If two chords, $AB$, $DE$, of a circle intersect at $C$, then

$$AC.CB = DC.CE \text{ (Fig. 9).}$$

If $A$, $D$, $B$, are the positions of the three posts, the error in taking $AC$ and $CB$ as 4·83 km each is negligible. (The diagram is obviously not to scale.) Similarly, $EC$ is the diameter of the earth less 1·83 m and may be taken as the diameter, $d$, with no appreciable error.

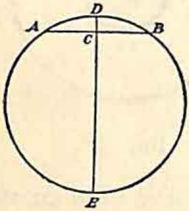


FIG. 9

$$4{\cdot}83 \text{ km} \times 4{\cdot}83 \text{ km} = d \times \frac{1{\cdot}83}{1000} \text{ km}$$

$$\therefore d = \frac{4{\cdot}83 \times 4{\cdot}83 \times}{1{\cdot}83} 1000 \text{ km}$$

$$= 12{,}748 \text{ km.}$$

It ought to be mentioned here that since the earth is not an exact sphere, its diameter varies between 12,756 km and 12,714 km. A mean value of 12,736 km however is accurate enough for most purposes.

## SUN AND EARTH

The movement of the earth with regard to the sun may be demonstrated by simple models. An effective method is to use a bright source of light (e.g. an electric lamp) for the sun and a white ball to represent the earth. The latter should not be larger than is necessary for it to be easily seen by pupils in all parts of the class-room. A long rod, such as a knitting needle, passing through the centre of the ball can represent the axis of the earth. The ends should protrude from the ball to show the position of the axis. Lines can be painted around the ball to indicate the equator, meridian of Greenwich, and any other required lines. If the room is in darkness apart from the 'sun', the light of 'day' and shadow of 'night' are clearly seen. More sharply defined shadows are obtained if the 'sun' is partly screened so as to prevent reflection from walls, ceilings, etc.

The orbit of the earth around the sun, and the seasons, are easily demonstrated. Care must be taken always to keep the axis tilted at the same angle to the plane of movement (Fig. 10).

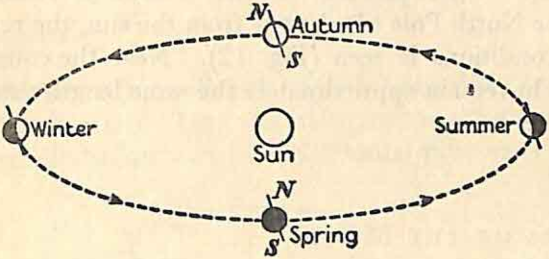


FIG. 10

## DAY AND NIGHT

The model also shows the varying lengths of day and night. Two parallels of latitude, equal distances north and south of the equator, are marked on the earth. The earth is placed in the summer position for the Northern Hemisphere, with the North Pole tilting towards the sun (Fig. 11). As the earth is rotated on its axis, a point on the north parallel

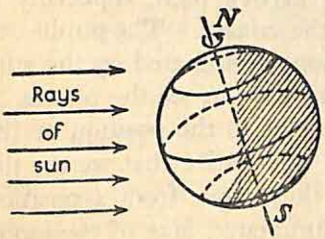


FIG. 11

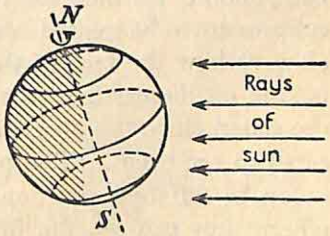


FIG. 12

of latitude is clearly seen to have a longer arc of travel in 'day' than in 'night', while in the Southern Hemisphere, 'night' is longer than 'day'. We can also observe that a place in the Southern Hemisphere may be in shadow while a position on the same meridian in the Northern Hemisphere is still lighted.

In the winter position for the Northern Hemisphere, with the North Pole tilted away from the sun, the reverse of these conditions is seen (Fig. 12). Near the equator, day and night remain approximately the same length throughout the year.

## PHASES OF THE MOON

The phases of the moon can be demonstrated in a similar fashion. The pupils must first realize that we see the moon because it is illuminated by the sun. The path of the moon around the earth can first be shown. The plane of this path is not the same as that of the earth's path around the sun, otherwise eclipses of the sun and moon would occur on every orbit of the moon. The plane of the moon's orbit is inclined to the orbit of the earth around the sun at an angle of about 5°. It is not necessary to rotate the moon around the earth at precisely this angle, but it is important that the path should be inclined to the earth's path, especially if eclipses are to be treated later in the course. The pupils can observe how the face of the moon, illuminated by the sun, is seen on the earth. It is not so easy to see the phases of the moon since the observer must be in the position of the earth to see them. The pupils will realize that we see the moon in different positions; if they move from a position where they can see the fully illuminated face of the moon (full moon), round to a position where they see the face in complete shadow, various phases of the moon will be seen on the way. (See also page 227).

Eclipses of the sun are shown by passing the moon between the sun and earth, so that the moon's shadow falls on the earth. Eclipses of the moon occur when the shadow of the earth falls on the moon.

## SUN-DIALS

The altitude type of dial has already been described in its simplest form (page 210). In another type the shadow can be a vertical shadow cast by a horizontal pointer (Fig. 13).

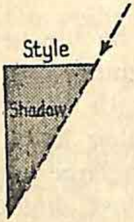


FIG. 13

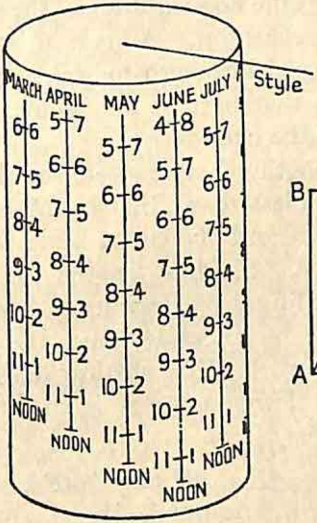


FIG. 14

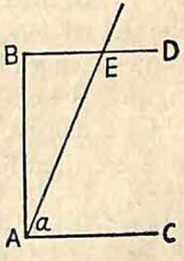


FIG. 15

This vertical form of altitude dial may easily be adapted to provide readings for each month of the year. The construction is shown in Fig. 14. The twelve shadow lines are spaced around a cylinder, each line labelled with the name of a month. The hours are marked along the lines according to the shadow lengths at these times. The shadow lengths are obtained in the same manner as those for the Egyptian dials, but the work is not quite so straightforward. The cylinder will probably be an empty tin of some kind. It should be as tall as possible so that the time scales may be widely spread. The maximum length of vertical shadow must, of course, be less than the height of the tin. The length of the style is chosen to fulfil this requirement; it may be found by drawing (Fig. 15). A line $AB$ is drawn equal to the height of the cylinder. Perpendiculars $AC$ and $BD$ are drawn at $A$ and $B$. $AE$ is drawn at an angle $\alpha$ to $AC$, where $\alpha$ is the noon altitude of the sun corresponding to its maximum elevation. A style of length $BE$ will then cast a shadow equal in length to $AB$. It should be noted that the style is that part of the pointer which protrudes over the edge of the cylinder.

The actual length of style used should be the nearest convenient length less than $BE$. If $BE$ is found to lie between, say, 8·4 cm and 8·6 cm a style length of 8·4 cm should be chosen. Shadow lengths for other hours and months are then found by drawing. A very large number of drawings will be necessary and the class should be grouped for this work. The shadow lengths can be calculated, of course, if trigonometry is known. From Fig. 15 it can be seen that

$$\begin{aligned} AB &= BE \tan \angle BEA \\ &= BE \tan \alpha, \end{aligned}$$

i.e., the shadow length is found by multiplying the style length by the tangent of the sun's altitude.

The hour lines are most easily set out on a sheet of paper, which is then wrapped round the cylinder. The width of the paper should equal the circumference of the cylinder, and the lines spaced evenly across it. When the dial is used, it is placed on a horizontal surface with the pointer placed to the appropriate month line and also directed towards the sun.

Fig. 16 shows one of the simplest forms of dial. A rod is fixed in a board and inclined at an angle equal to the latitude of the place where the dial is to be used. On the rod, and perpendicular to it, is placed a circular dial marked in hours at 15° intervals. These should be marked on both sides of the dial since the shadow will sometimes be on the upper surface, sometimes on the lower surface. The style must incline towards the north, and the noon hour line must also point in that direction.

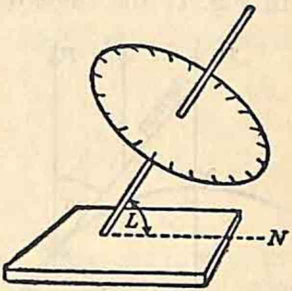


FIG. 16

## STARS AND PLANETS

Stars and planets must be distinguished from one another. Those bodies which, like the earth, revolve around the sun are the planets. Five are visible to the naked eye—Mercury, Venus, Mars, Jupiter, and Saturn: others can be seen through a telescope. The planets are comparatively close to the earth. Saturn, the farthest of the bright planets, is about 128 crores kilometres away. The nearest of the fixed stars, Proxima Centauri, is about 40 billion crores kilometres away. The stars are bodies similar to the sun. Because of the rotation of the earth they appear to rotate around the Pole Star. The apparent movements of the stars are regular, but the observed paths of the planets do not follow the same pattern. (The word *planet* comes from the Greek word meaning *wanderer*.) Thus, although the planets appear to us to move among the stars, in actual fact we are observing their motion against a distant background of stars.

*To find latitude from the stars.* The simplest method of determining latitude is from observation of the Pole Star. In Fig. 17 the angle of elevation of the Pole Star is 40° when observed from a point $A$, whose latitude is $L$.

$\angle CAB + \angle CAH + \angle HAP = 180°$.

But $\angle CAH = 90°$ (angle between radius and tangent).

$\therefore \angle CAB + \angle HAP = 90°$.

But $\angle CAB + \angle ACB = 90°$ (two smaller angles of a right-angled triangle).

$\therefore \angle ACB = \angle HAP$,

i.e. $L = 40°$.

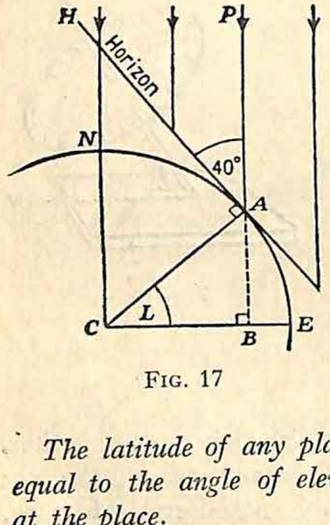


FIG. 17

*The latitude of any place in the Northern Hemisphere is equal to the angle of elevation of the Pole Star measured at the place.*

Latitude can also be found by measuring the altitude of any other star at its transit. For stars seen by an observer facing north, the latitude, $l$, is given by :

$l = d - (90° - \measuredangle)$ where $d$ = declination of star
$\measuredangle$ = altitude of star at transit.

For stars seen by an observer facing south the latitude is given by

$$l = d + (90° - \measuredangle).$$

e.g. *An observer on a ship, facing north, measures the altitude of Capella at transit as 58° 30′. Find the latitude of the ship.*

The declination, $d$, of Capella is 45° 57′ (from a nautical almanac)

$$\therefore\ l = 45^\circ\ 57' - (90^\circ - 58^\circ\ 30')$$
$$= 14^\circ\ 27'\ \text{N}.$$

A positive value for $l$ indicates a latitude North; a negative value a latitude South.

## THE STARS: DISTANCES

Very large numbers are involved when measuring the distances of the stars in miles or kilometres so these distances are usually reckoned in *light-years*. A light-year is the distance travelled by light in a year. Light travels at 2,99,793 kilometres per second, so a light-year is

$$2{,}99{,}793 \times 60 \times 60 \times 24 \times 365 \text{ km}$$
$$= 9{\cdot}45 \times 10^{12} \text{ km (approx.)}$$

The distance of the nearer stars can be found by taking observations of the star from the earth, when the latter is at different points on its orbit around the sun. Since the radius of this orbit is $149{\cdot}9 \times 10^6$ kilometres, a base line of $299{\cdot}8 \times 10^6$ kilometres is obtained, if readings are taken at opposite points on the orbit. The second reading will be taken six months after the first reading. It is usual, however, to take readings at points which are separated by $149{\cdot}9 \times 10^6$ kilometres, i.e. a distance equal to the radius of the earth's orbit. Even with so large a base line the difference in the directions of the star is very small. In Fig. 18 $E_1$ and $E_2$ are two positions of the earth, where $E_1 E_2 = 149{\cdot}9 \times 10^6$ kilometres. For the nearest star, Proxima Centauri, the difference between the directions of $S$ is only 0·75 seconds or about $\frac{1}{5000}$ of 1°. (This is approximately the angle subtended by a line 2·5 cm long at a distance of 6·5 km). The difference in readings is equal to the angle subtended by $E_1 E_2$ at $S$, i.e. $\angle E_1 S E_2$. This angle is known as the parallax of the star. With so small an angle the error in taking the base line, $E_1 E_2$, as part of a circle, centre $S$, is negligible.

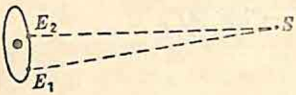


FIG. 18

Hence,
$$\frac{149{\cdot}9 \times 10^6}{2\pi E_2 S} = \frac{0{\cdot}75}{360 \times 60 \times 60}$$

or $E_2 S = 41 \times 10^{12}$ kilometres approximately.

The above calculation is easier if radian measure is used. The distance of the star, $d$, is given by

$d = \dfrac{r}{\theta}$ where $r = E_1 E_2 =$ radius of earth's orbit.
$\theta =$ angle subtended in radians.

These methods can only be used for the nearer stars; during the present century new methods using spectroscopy have been developed for finding the distances of the farther stars.

## THE PLANETS

The planets rotate around the sun in orbits which lie, almost in the same plane. The laws which govern their movement around the sun were stated by Kepler in the early seventeenth century after more than nine years of calculation. They are:

1. Every planet moves in an ellipse with the sun at one focus.

2. The straight line joining the planet to the sun (the *radius vector*) sweeps out equal areas in equal times.

3. The square of the time taken by a planet to revolve around the sun (i.e. its 'year') is proportional to the cube of its average distance from the sun.

In most cases the orbits of the planets approximate to circles; the earth's orbit, for example, is an ellipse with axes whose lengths are in the ratio 7,001 : 7,000.

| | *Mercury* | *Venus* | *Earth* | *Mars* | *Jupiter* | *Saturn* |
|---|---|---|---|---|---|---|
| Average radius of orbit (millions of miles) | 36<br>(0·4) | 67<br>(0·7) | 93<br>(1·0) | 142<br>(1·5) | 484<br>(5.2) | 887<br>(9·5) |
| Length of year | 88 days | 225 days | 365¼ days | 687 days | 12 years | 30 years |
| Rate of rotation round the sun (degrees per day) | 4·1° | 1·6° | 1° | 0·52° | 30° per year | 12° per year |

The numbers in brackets underneath the radii compare them with the radius of the earth's orbit.

## THE INNER PLANETS

Two planets, Mercury and Venus, lie nearer the sun than does the earth. From Fig. 19 it should be clear that an inner planet can only be seen as a morning or evening star. It can never be seen at night when the observer's position on the earth is turned away from the sun. Assuming that we view from a point above the

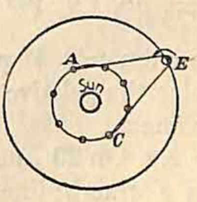


FIG. 19

North Pole, the earth rotates anticlockwise on its axis. An inner planet in position $A$ will be visible to an observer before he can see the sun, i.e. it is a morning star. In position $C$, the star remains in view after the sun has set; it is an evening star.

It will be seen from the table that the distances of the earth, Venus, and Mercury from the sun are approximately in the ratio 10 : 7 : 4. Their paths can be represented therefore by circles with radii in the same ratio, e.g. 10, 7, and 4 centimetres. The earth moves around the sun at the rate of $\frac{360^\circ}{365}$, or approximately 1° per day; Venus at the rate of $\frac{360^\circ}{225}$ or 1·6° per day; Mercury at the rate of $\frac{360^\circ}{88}$ or approximately 4° per day. This information enables us to plot the relative positions of the earth and the two inner planets.

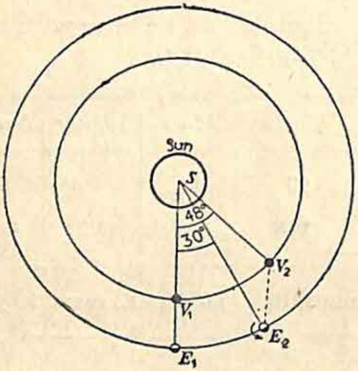


FIG. 20

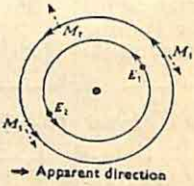


FIG. 21

e.g. On 20 June 1952, the earth, Venus, and the sun were in a straight line, $SV_1\ E_1$ (Fig. 20). On 20 July, 30 days later, the earth will have moved 30° (1°×30). Venus has moved 48° (1·6°×30). Their positions are shown at $E_2$ and $V_2$. If the figure is drawn to scale, the transit time of Venus can be determined from the angle $SE_2V_2$.

From the diagram $\angle SE_2V_2 = 33°$. To an observer on the earth, Venus will appear 33° to the right of the sun. It will cross the N. – S. line 33 × 4 minutes before the sun,

i.e. 2 hr. 12 min. before the sun.
i.e. at 9.48.

We can also see that Venus is a morning star on 20 July.

## THE OUTER PLANETS

The earth is rotating at a greater angular velocity than the outer planets. To an observer on the earth at $E_1$ (Fig. 21), the planet Mars at $M_1$, although travelling in the same direction, will appear to be travelling in the opposite direction because of the greater velocity of the earth. Six months later, when the earth is at $E_2$, Mars is at $M_2$. It will appear to have changed direction to an observer on the earth. Later still the earth is at $E_1$ again, and Mars will be at $M_3$. Once more it will appear to have changed its direction. Thus the outer planets appear to move backwards and forwards among the fixed stars.

The transit times of the planets are given for each day in a nautical almanac. It is possible to use these to find the relative positions of the earth and a planet (Fig. 22).

e.g. On 4 July the transit time of Jupiter was 8.00. Draw the orbits of the earth and Jupiter to scale (ratio 1:5·2, see table on page 223). Radii of 1 cm and 5·2 cm, or 2 cm and 10·4 cm, could be used. Place the sun at the centre, $S$ and the earth at any point, $E$. Join $SE$.

The transit time of Jupiter is 4 hours before noon.

$\therefore$ Jupiter is $15^\circ \times 4 = 60^\circ$ to the right of the sun.

Construct $\angle SEJ = 60^\circ$ to cut orbit of Jupiter at $J$.

$J$ is the position of Jupiter.

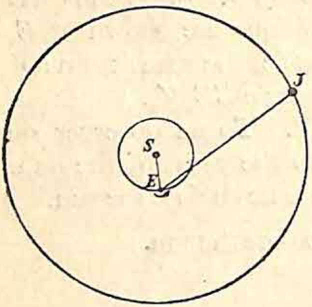


FIG. 22

## THE MOON

The day and the year are both vitally important units of time, depending upon the apparent motion of the sun. Nevertheless, many of the early calendars were based upon the phases of the moon. In our own calendar, the date of Easter depends upon the state of the moon.

The moon rotates around the earth in 27 days 8 hours. After this period of time it will be seen in the same position relative to the stars. The phases of the moon, however, depend on its position relative to the sun. During the $27\frac{1}{3}$ days the earth has moved along its path round the sun, and it takes $29\frac{1}{2}$ days for the moon to return to the same position relative to the sun. This is the period of one new moon to the next new moon.

## APPARENT SHAPE OF THE MOON

The moon is commonly referred to as the new moon when the crescent shape appears after a few days during which no moon has been visible. This slender crescent might better be named 'the young moon', for the time of the new moon is when the moon presents its shadowed face and is therefore invisible (Fig. 23). It is the opposite of full moon.

Since the moon completes one cycle in $29\frac{1}{2}$ days, its movement per day is $\frac{360^\circ}{29\frac{1}{2}} = 12^\circ$ approximately. Each day the moon will appear 12° from its position at the same time the previous day.

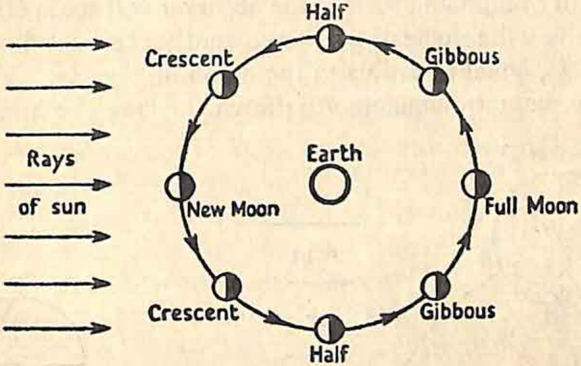


FIG. 23

The shape of the moon at any date may be found by drawing.

e.g. *Draw the shape of the moon on 23 December 1971.*

The new moon occurred on 17 December.

∴ On 23 December the moon was 6 days old.

It will have moved $12^\circ \times 6 = 72^\circ$ from the new moon position (Fig. 24). In the new moon position the line, *EN*, joining the earth to the moon is parallel to the rays of the sun. A line, *EM*, at 72° to *EN*, fixes the position of the moon on 23 December. There is no need to draw the moon and its distance from the earth to scale, but it should be remembered that in the diagram the size of the moon is grossly exaggerated.

Through $M$, the centre of the moon, a diameter, $AB$, is drawn perpendicular to $EM$. An observer on the earth will see the moon in the direction $EM$, and part of the surface facing him will be in shadow. A line $DC$ is drawn parallel to $EM$ and passing through the shadow edge $D$. The greatest width of illuminated surface the observer will see is $CB$.

To draw the shape of the moon, another circle is described (Fig. 25), equal in radius to the moon of Fig. 24. Vertical and horizontal diameters are drawn. Along the horizontal

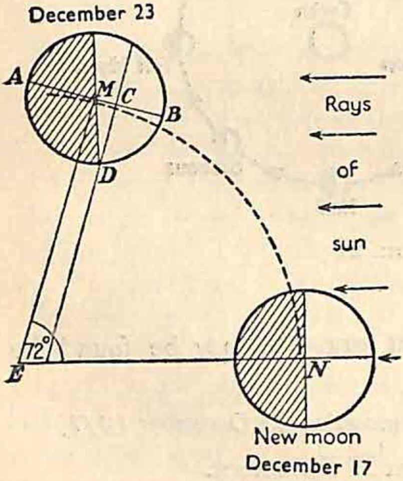


FIG. 24

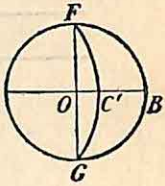


FIG. 25

diameter a distance $C'B'$ is marked, equal to $CB$ of Fig. 24. The circle represents the moon with $C'B'$ the width of its visible crescent. The shape of the moon can be roughly drawn through the points $F$, $C'$, $G$, $B'$. A more accurate construction is possible if it is realized that $FC'G$ is half an ellipse with semi-major axis $OF$, and semi-minor axis $OC'$. The method of auxiliary circles (page 251) is suitable for constructing the ellipse.

## LUNAR DATES

If the date of the full moon is known for one month, the dates of other full moons, or other phases of the moon, may be found.

e.g. Full moon occurred on 11 March 1971.

The following full moons were 10 April, 10 May, 8 June, etc. (29 or 30 days are added alternately). The new moon occurs approximately 15 days before and after full moon, e.g., a new moon occurred on 24 April 1971.

Since the lunar month is 29½ days, the old civilizations, who based their months upon the phases of the moon, generally made them alternately 29 and 30 days long. The twelve lunar months amount to 354 days, or 11 days less than the normal year. The date of any particular phase of the moon becomes 11 days earlier each year, e.g. the full moon in November 1970 occurred on 13 November; in 1971 it occurred on 2 November 1971.

The 'age' of the moon at any time is the number of days since the last new moon. e.g. A new moon occurred on 17 December 1971.

On 18 December the moon was 1 day old
,, 19 ,, ,, 2 days ,,
,, 20 ,, ,, 3 ,, ,,

For each succeeding year the age of the moon at any date is 11 days greater.

The lunar month of 29½ days is not an exact sub-multiple of the solar year of 365¼ days. The Greek astronomer, Meton (450 B.C.), established the link between the two periods by observing that in 19 years there are 235 lunar months. The dates of full moons, etc., pass through a cycle which is repeated every 19 years. It is said that Meton had the years of the cycle inscribed in gold on a Greek temple, and it is from this that we get the term *Golden Number* for the number of a year in the cycle. The present rule for finding the Golden Number of a year is: *Divide the A.D. date by 19. The Golden Number is the remainder plus 1.* e.g. The remainder when 1971 is divided by 19 is 14. The Golden Number for 1971 is $14+1=15$. All years with Golden Numbers 15 will have the dates of full moons, etc., the same as in 1971.

## XIV

# DRAWING: CURVE STITCHING

PRACTICAL drawing is taught in most schools for boys and girls. The value of drawing can be considerable, provided the lessons go beyond the mere techniques of drawing. Often the work done is a careful copy of a drawing from a book, or from the blackboard. This may encourage accuracy and neatness, but these are only two of the assets of a good drawing course, and opportunity is being wasted.

An important feature of practical drawing is the training it can give in three-dimensional visualization. We live in a space of three dimensions, yet the majority of people, when asked to visualize a solid object, imagine a flat picture of the object. They find great difficulty in thinking in terms of depth. If we are to attack problems in space satisfactorily we must be able to think in terms of three dimensions. Practical drawing is one of the most valuable methods of training the mind for this purpose.

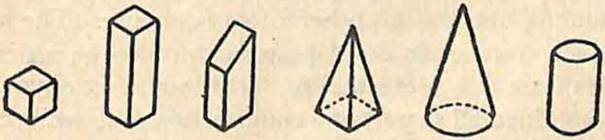


FIG. 1

It will readily be seen that copying drawings can help little in this respect. It is essential that solid objects be used, handled, and drawn during the course. A set of wooden solids may be acquired. A few suggested shapes are shown in Fig. 1.

An alternative to the wooden blocks is a set made from drawing-paper or thin card. The making of such models is a useful exercise for pupils (page 245). The better models could be kept each year to augment the collection.

Skeleton models (Fig. 2) made from rods or knitting needles, joined at their ends by corks or small lumps of modelling clay, are also useful. They are especially valuable for seeing sections (page 243), or other views obstructed in the normal solid models.

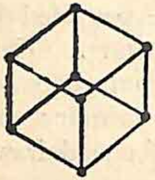


FIG. 2

An adequate supply of specimens must be available so that two or three blocks can be grouped to increase the difficulty of the drawing. Two pupils, or four if necessary, can share a group of blocks for an exercise.

In schools the practical drawing course may be linked with the woodwork or handicraft. This is excellent in the early days, but it is a pity if the link is too close since it unduly restricts the range of drawing done.

The following notes indicate briefly types of work in drawing, most of which will appeal to both boys and girls. It is not necessary to teach all the forms of drawing shown. The simpler pictorial projections can be taught separately, without reference to any other form of drawing. The making of solids from their developments can also be an isolated piece of work. Nevertheless, great value is obtained by treating them all as part of a comprehensive drawing course.

## PROJECTIONS

The various ways of drawing objects are known as projections. The type of projection in most common use is orthographic projection, but it is not the simplest, nor is it the most interesting to start with. The disadvantage of orthographic projection is that it does not present a pictorial view. Moreover, it is only the trained eye that can visualize an object from its orthographic projection.

## PICTORIAL PROJECTIONS

There has been considerable development within the last twenty years of types of projection which enable the untrained person to grasp the shape and details of an object. The value of such drawings in large-scale production is obvious.

Most of the forms of pictorial projection are easy to understand, and not difficult to draw when certain fundamentals have been grasped. It is this branch of drawing which is especially recommended to girls' schools. Girls generally enjoy drawing, and these forms of projection are often used to illustrate items which are of interest to girls: the appearance of a room; the arrangement of furniture in a room or kitchen; the details of a kitchen fitment, etc. The simpler forms of pictorial projection present views of objects, but they do not give perspective views. Hence there is often a feeling that there is something wrong about such a projection. They are not intended to be pictures as seen by the eye, but pictures drawn according to certain rules which greatly simplify the drawing problems and, at the same time, enable an impression of the object to be obtained easily by the unskilled observer.

In the drawing lesson high standards should be set and expected from the very beginning. The pupils should see good drawings which will encourage them in their own efforts. Pictorial projections are commonly used in building, etc., and such drawings can often be found in advertisements of kitchen furniture, or in the pages of those periodicals devoted to the house. Rough sketching may be used to help the child to see the problems of a particular drawing. This work is made easier by the use of 0·5 cm squared paper. The lines provide a 'grid' upon which the child constructs his drawing. It can be used for 'oblique' sketches (Fig. 3). If the paper is turned so that the lines are at 45° to the horizontal, it presents an 'axonometric' grid (Fig. 4). It is possible to buy paper with isometric ruling, but the normal squared paper should cover most requirements.

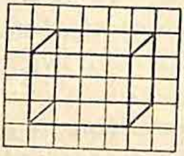


FIG. 3

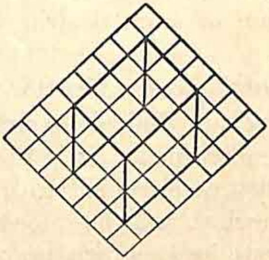


FIG. 4

Whatever forms of drawing are done, however short or long the drawing course may be, conventional practice (page 246) should be followed. It is absurd to accustom a pupil to draw centre or dimension lines, hidden details, etc., in a manner not used by drawing-offices. His ability to read standard drawings will obviously suffer.

(*a*) *Oblique projection* (Fig. 5). This is probably the easiest form of projection with which to commence a course. A facing view of one side is drawn full scale, or to some convenient scale. The lines which are perpendicular to this face in the object are shown receding at 45° in the projection. 45° is the value most commonly used, but any other angle may be substituted. Different values show more clearly different aspects of the object; pupils should be encouraged to experiment with different angles.

The receding lines are generally drawn to half the scale used for horizontal and vertical lines. This avoids giving the object an appearance of greater depth than it actually possesses.

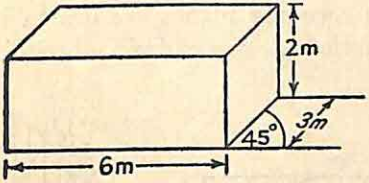


FIG. 5

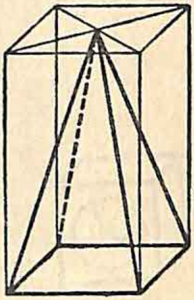


FIG. 6

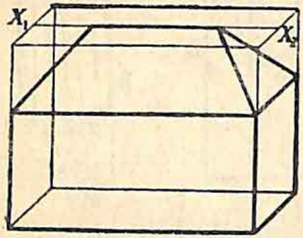


FIG. 7

Rectangular solids, or combinations of rectangular solids, are quite simple to draw in oblique projection. The easiest way to deal with other solids is to draw the rectangular framework which contains them. (This is sometimes referred to as the 'crate', and the process is known as 'crating'.) Thus, the pyramid in Fig. 6 is drawn by first constructing the rectangular framework or 'crate' which contains it. The apex of the pyramid is found by the intersection of diagonals, or of lines joining the midpoints of opposite sides. The simple outline of a house in Fig. 7 is dealt with in a similar fashion. The ridge is found by joining midpoints, $X_1$, $X_2$, and marking off the correct distances along it.

Oblique projections of curved lines and surfaces present some difficulty and this part of the work should be left to a later stage. Circular lines on facing planes remain their true shape (Fig. 8), but those on receding planes are drawn in the following manner: The circle is drawn and placed within a square framework (Fig. 9) which it touches at *A*, *B*, *C*, and *D*. A number of vertical lines, *EF*, *GH*, etc., are drawn, cutting the circle at *P*, *Q*, *R*, *S*, etc. The oblique view of the square is then drawn (Fig. 10). *A′*, *B′*, *C′*, and *D′* will remain the midpoints of the sides, but different scales will be used for the vertical and for the receding lines. The lines *E′F′*, *G′H′*, etc., are drawn, and the positions of the points, *P′*, *Q′*, *R′*, *S′*, etc., found by marking off *E′P′* = *EP*, *F′Q′* = *FQ* etc. The curve is then drawn freehand to pass through the plotted points.

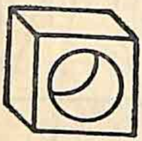


FIG. 8

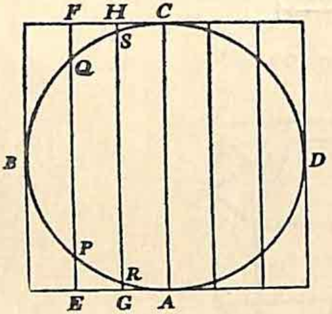


FIG. 9

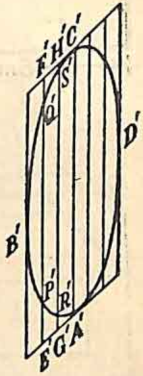


FIG. 10

Any curves may be drawn in oblique view by the above method of ‘ordinates’. It is, of course, quicker to draw facing views of curves, and these generally present better views.

(*b*) *Isometric projection* (Fig. 11). In isometric projection both vertical faces of a rectangular block are shown receding at angles of 30°. Lines in an isometric projection should properly be drawn according to an isometric scale, but for most purposes, and certainly in schools, this can be disregarded. If further information about isometric scale is desired, any of the standard textbooks on practical drawing should be consulted.

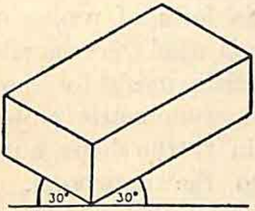


FIG. 11

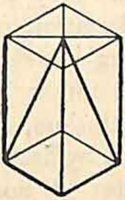


FIG. 12

The difficulties of isometric projection are similar to those of oblique projection, and can be dealt with in the same manner. The rectangular framework is drawn first when dealing with pyramids, etc. (Fig. 12), and circles and curved lines are drawn by a method similar to that shown on page 236.

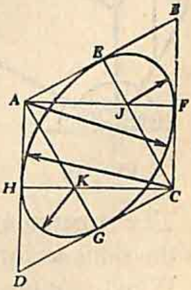


FIG. 13

The time taken to draw circles, etc., in isometric views can be quite considerable, especially since no facing view is possible as in oblique projection. A quick method of drawing approximate 'ellipses' is shown in Fig. 13. As before, the circle is contained within a square framework *ABCD*, whose isometric view is drawn. The midpoints of the sides, *E*, *F*, *G*, *H* are found. The lines *AF*, *AG*, *CE*, *CH* are drawn, and intersect at *J* and *K*. Arcs of circles with centres *A*, *C*, *J*, *K* are described as shown. It must be realized that this is a non-mathematical method and gives only an approximation to the true shape.

(*c*) *Axonometric projection.* This form of projection is becoming increasingly popular; it is used very largely by builders and architects as it is especially useful for showing room interiors, fitments, etc. An axonometric projection is started by first drawing the plan in its true shape, but with the sides at some inclination to the horizontal. The pairs of angles most commonly used are 30°, 60° (Fig. 14) and 45°, 45° (Fig. 15).

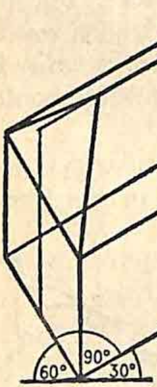


FIG. 14

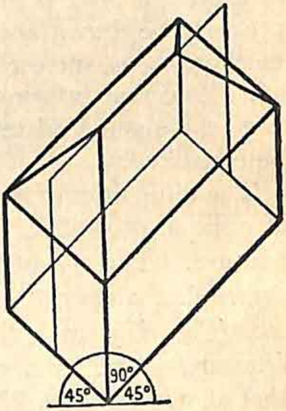


FIG. 15

The treatment of non-rectangular solids, curves, etc., is the same as for oblique and isometric projections.

When some skill has been acquired with pictorial projections, they can be used for showing arrangement of furniture in a room, details of fitments, etc. Figs. 16-18 show three pictorial views of part of a kitchen.

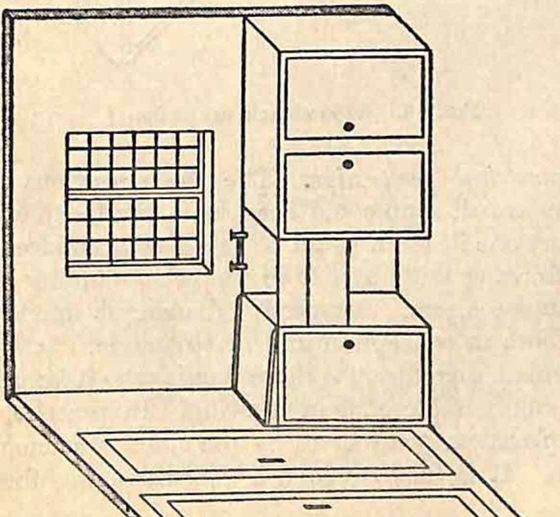


FIG. 16. Oblique projection

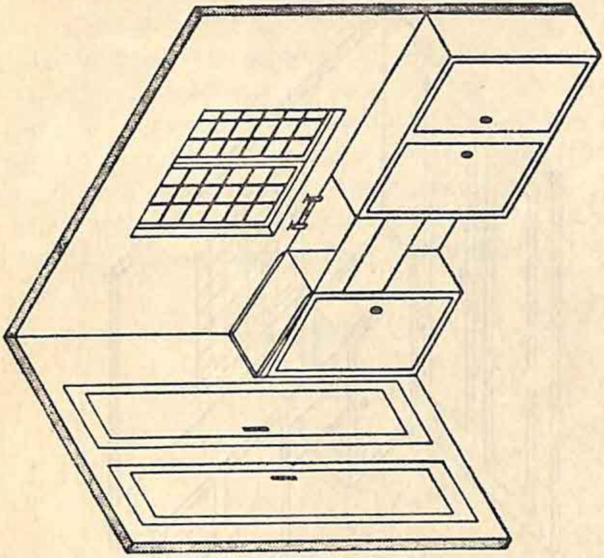


FIG. 17. Isometric projection

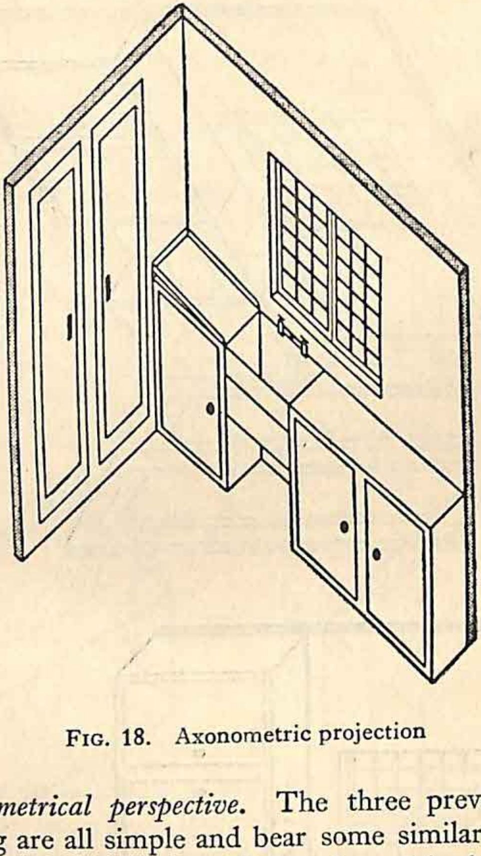


FIG. 18. Axonometric projection

(*d*) *Geometrical perspective.* The three previous forms of drawing are all simple and bear some similarity to each other. Any one of them could be taken with children who have no drawing skill apart from the general ability to use ruler, compasses, etc. Perspective drawing is much more difficult, both in conception and in technique. It is only recommended, therefore, for those pupils who have already acquired considerable skill in drawing. In practice, it is being displaced in many cases by the simpler axonometric projection. It is, nevertheless, a very satisfying form of drawing, and it demands much from the pupil in the way of three-dimensional visualization. It might well be regarded as the culmination of a good drawing course.

Some demonstration of its meaning and form must first be given. One noteworthy method is to stand at arm's length from a window, and sketch on the window with wet chalk the outlines of a hut, or other object, as seen through the window (Fig. 19). In the finished sketch, each line on the drawing will cover the corresponding line on the object.

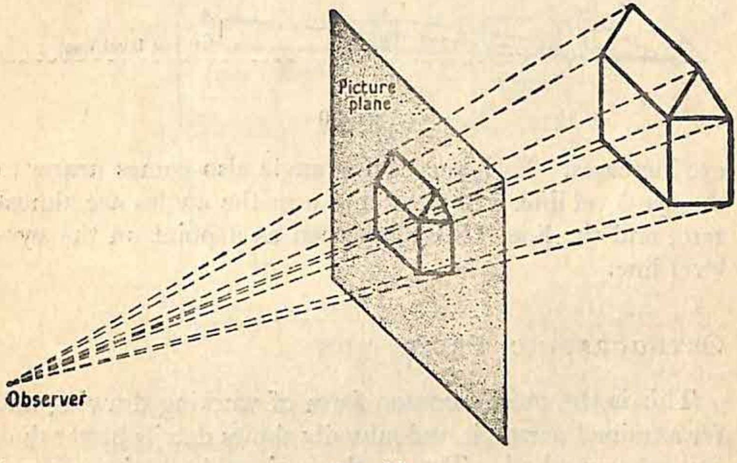


FIG. 19

It will be seen that we have shown on a flat surface the eye's view of a solid object. The flat surface on which the drawing is made is known as the *picture plane.* In the above example the picture plane is in front of the object. This is its usual position, but it may be placed behind or passing through the object.

In the forms of projection already described, lines which are parallel in the object remain parallel in the drawing. In simple perspective drawing, lines which are parallel to the picture plane remain parallel to each other in the drawing. Each other set converges to a vanishing point. This conforms to the impression obtained by the eye; railway lines, or the sides of a long, straight road appear to converge to a vanishing point situated at eye level. This can be explained if we first realize that the apparent size of an object as seen by the eye depends on the angle subtended at the eye. In Fig. 20 the angles subtended by the equal lines, $A_1B_1, A_2B_2, A_3B_3$, decrease as their distance from the eye increases. Each succeeding angle also comes nearer to the eye-level line. At great distances the angles are almost zero, and the line $AB$ is then seen as a point on the eye-level line.

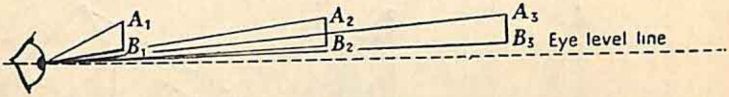


FIG. 20

## ORTHOGRAPHIC PROJECTION

This is the most common form of working drawing and for a trained person it undoubtedly shows details better than any other method. Two or three views (sometimes more) of the object are shown, the views being taken from directions perpendicular to each other.

The views taken in horizontal directions are elevations (front and side). A view taken from above is the plan. The three views are positioned as shown in Fig. 21. It should be noted that the plan, which is observed from *above* the object, is placed *below* the front elevation. The side elevation obtained by viewing from the *left*-hand side is placed on the *right* of the front elevation. This is common British practice and is known as first-angle projection.

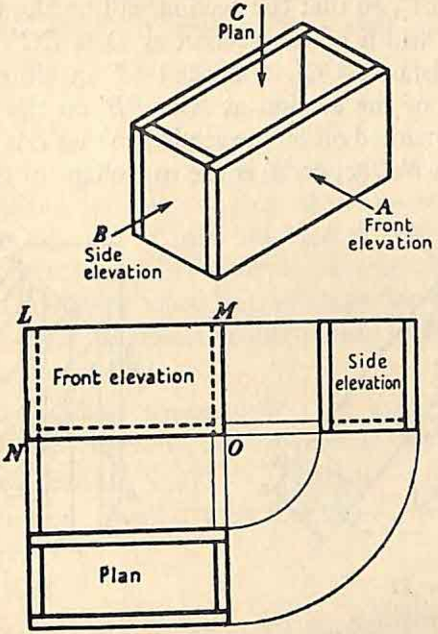


FIG. 21

(*a*) *Sections.* The drawing of the true shape of a section from orthographic views can be an excellent drawing exercise.

The elevation and plan of a pyramid is shown in Fig. 22. If the pyramid is cut by a plane, *AA*, and the top part removed, the cut face is known as a section. The true shape of the section is found in the following way:

1. The plan view of the section is drawn: lines from *B* and *C* are projected to the plan to give the lines of cutting, *BB′*, *CC′*, on two opposite faces of the pyramid. *BC* and *B′C′* will be the cutting lines on the other two faces.

2. Lines perpendicular to *BC* are projected to some convenient place on the paper. A centre line, *DE*, is drawn

parallel to *BC*, so that the section will be shown in its true size. The width of the section at *D* is *CC′* on the plan. Half the distance *CC′* is marked off on either side of *D*. The width of the section at *E* is *BB′* on the plan. This distance is marked off on the section so that *E* is its midpoint.

3. Draw *b′c′*, *bc*; *bcc′b′* is the true shape of the section.

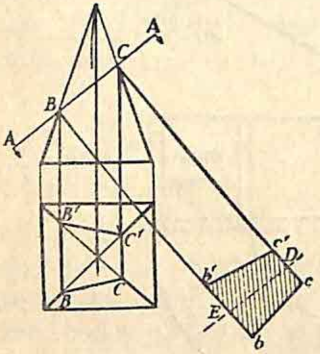


FIG. 22

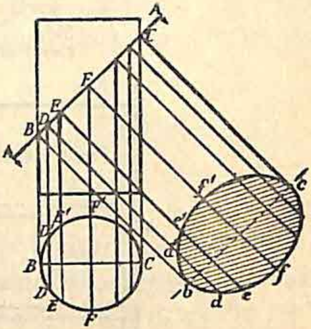


FIG. 23

It will be seen that, to draw the section, information is obtained from elevation and plan in conjunction. This is valuable training in reading orthographic projections, and in solid visualization.

Sections of solids with curved surfaces demand a slight extension of the method given above. Fig. 23 shows the construction of a section of a cylinder.

The plane, *AA*, cuts the cylinder at *B* and *C*. A number of points, *D*, *E*, *F*, etc., are selected along *BC*. (The centre of *BC* should be used as one of these points). Lines are projected from *D*, *E*, etc., to the plan. *DD′* on the plan gives the width of the section at *D; EE′* the width at *E*, etc.

A centre line, *bc*, for the section is drawn as before, and lines projected to it from *D*, *E*, etc. The width of the section *dd′*, is taken from *DD′* on the plan. The points *d*, *d′*, *e*, *e′*, etc., are plotted in this way. Finally a freehand curve is drawn through them; it will be seen that an ellipse is obtained.

### DEVELOPMENTS

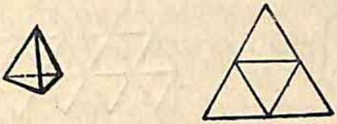
A development or net of an object is a two-dimensional figure which gives the entire surface of the object. In some cases the development is best obtained from an elevation and plan, but a large number of developments can be drawn without any knowledge of orthographic projection. Much interesting work can be done in constructing solids from thin card or drawing-paper.

The simplest solids to construct are those sometimes known as the Platonic solids, the five regular solids known to the Greeks. They are:

(*a*) *Tetrahedron:* a solid with four faces, each an equilateral triangle (Fig. 24).

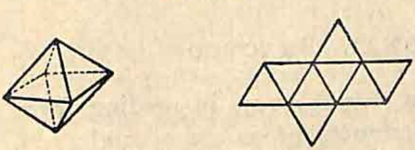
(*b*) *Cube.*

(*c*) *Octahedron:* eight faces, each an equilateral triangle (Fig. 25).



Tetrahedron Net of tetrahedron

FIG. 24



Octahedron Net of octahedron

FIG. 25

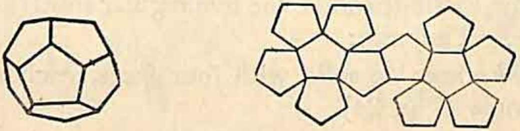
(*d*) *Dodecahedron:* twelve faces, each a regular pentagon (Fig. 26).

(*e*) *Icosahedron:* twenty faces, each an equilateral triangle (Fig. 27).

If the faces (F), vertices (V), and edges (E) of the regular solids are counted it will be found that they are related by the formula

$$F + V = E + 2.$$

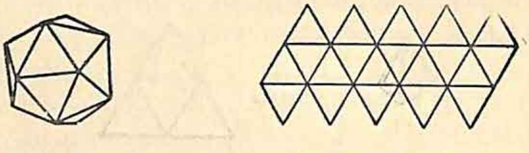
This result is due to Euler, the blind French mathematician. (A comprehensive treatment of these and other solids is given in *Mathematical Models* by Cundy and Rollett, O.U.P.).



Dodecahedron

Net of dodecahedron

FIG. 26



Icosahedron

Net of icosahedron

FIG. 27

## CONVENTIONAL PRACTICE

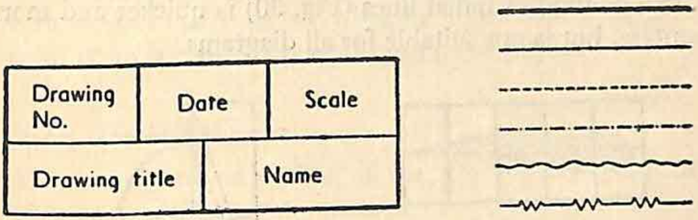
In order to reduce errors in reading from drawings the following recommendations are offered:

(*a*) *Title block.* On each drawing, the following information should be placed in a block in the bottom right-hand corner of the sheet: Drawing number, title of drawing, date, scale, name of draughtsman.

(*b*) *Lines.* Thick continuous lines are used for visible outlines. Thin continuous lines are used for dimensions and projection lines. Short dashes indicate hidden detail.

Alternate long and short dashes are used for centre lines also for showing path of movement. Centre lines should project beyond the view shown.

Break lines are used to shorten views of long, uniform objects.



Title block

Types of lines

FIG. 27

(*c*) *Sections.* The position of a section is shown on one or more views of the object by letters and arrows (see Fig. 22). Sectional views are indicated by hatching (thin lines sloping at 45°).

(*d*) *Dimensions.* Dimension lines should lie clear of the drawing as far as possible, and be placed below or to the right (Fig. 28). The dimension is written perpendicular to the dimension line, so that all dimensions can be read from below, or from the right-hand side of the drawing. The longer dimension lines should lie outside the shorter ones. Projection lines from the drawing indicate the exact end of each dimension line.

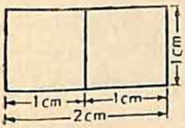


FIG. 28

## Reduction and Enlargement

Among the topics which can profitably be studied in drawing is that of reduction and enlargement of drawings or diagrams. The simplest method to follow is the method of squares (Fig. 29), which is most useful when the diagram to be copied is curved or irregular. A grid of squares is placed over the diagram. Another similar grid is drawn to the new size desired. The diagram is then copied square by square in the new grid.

The method of radial lines (Fig. 30) is quicker and more accurate, but is not suitable for all diagrams.

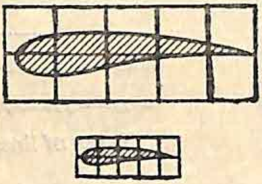


FIG. 29

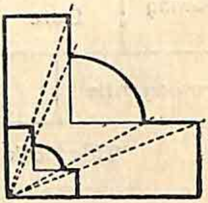


FIG. 30

A common mechanism for enlarging or reducing is the pantograph (Fig. 31); it can be made from Meccano parts.

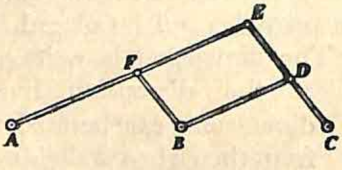


FIG. 31

A freely moving linkage is made with $BD = FE$, and $FB = ED$, so that $FBDE$ is a parallelogram. $A$, $B$, $C$

must lie on a straight line; this condition is fulfilled if the length of *DC* is such that

$$\frac{DC}{BD}=\frac{FB}{AF}.$$

*A* is kept fixed, and *B* is moved so as to trace a diagram; a marker at *C* traces the diagram to an enlarged scale. (The triangles *AFB*, *AEC* are similar for all positions of the mechanism.) Any one of the points *A*, *B*, *C* may be fixed, the other two will trace similar figures. The enlargement or reduction scale is altered by varying the proportions of the linkage. There is an obvious opportunity here for individual experiment by small groups of pupils.

## CONIC SECTIONS

The conic sections are the ellipse, the parabola, and the hyperbola. They may be obtained by cutting a cone in different ways (Figs. 32-34). A study of them is generally

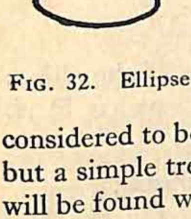


FIG. 32. Ellipse

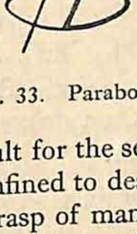


FIG. 33. Parabola

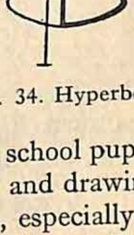


FIG. 34. Hyperbola

considered to be too difficult for the secondary school pupil, but a simple treatment confined to description and drawing will be found within the grasp of many pupils, especially if the sections are demonstrated by cutting cones of modelling clay with a sharp knife or razor blade. A particular fascination lies in the different and apparently unrelated constructions by which they may be drawn. Some of these constructions are given below.

### ELLIPSE

1. Most of the drawing constructions of the conic sections are the plotting of loci; one of the simplest of these is the loop method of drawing ellipses. An ellipse is traced by a point which moves so that the sum of its distances from two fixed points is constant. Two pins are placed in a sheet of paper, with a loop of cotton or fine string around them (Fig. 35). The point of a pencil is inserted within the loop so that the cotton is stretched tight; its path is an ellipse.

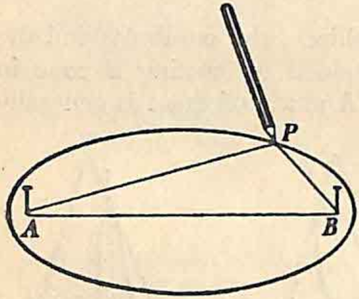


FIG. 35

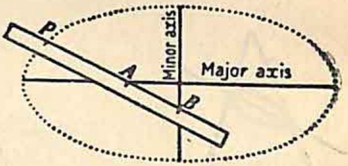


FIG. 36

2. *Trammel.* The major and minor axes of the ellipse are drawn (Fig. 36). On a strip of paper three points, $P$, $A$, and $B$, are marked with:

$$PA = \tfrac{1}{2} \text{ minor axis}, \quad PB = \tfrac{1}{2} \text{ major axis}.$$

This strip is known as a trammel. It is placed so that $A$ lies on the major axis, and $B$ lies on the minor axis. $P$ will then lie on the perimeter of the ellipse. A series of positions for

*P* are found and marked, moving *A* and *B* along the axes. The ellipse is then drawn freehand through these points.

3. *Auxiliary circles.* The major and minor axes are drawn as before. With their point of intersection as centre, two concentric circles are described, with diameters equal to the major and minor axes (Fig. 37). A line from the centre, *O*, cuts one circle at *A*, the other at *B*. From *A* and *B*, the lines *AP*, *BP* are drawn parallel to the major and minor axes respectively. Their point of intersection, *P*, will lie on the ellipse. A number of similar points are found by drawing a series of radial lines from *O* (only one is shown in Fig. 37). The ellipse is drawn through these points.

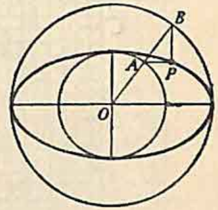


FIG. 37

4. *Paper folding.* A circle is drawn and cut out; a point *P* is marked inside it (Fig. 38). The circle is folded so that the circumference passes through the point *P* (Fig. 39). This is repeated several times, and it will be found that the creases form the envelope of an ellipse (Fig. 40).

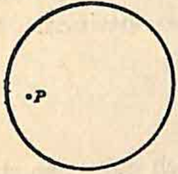


FIG. 38

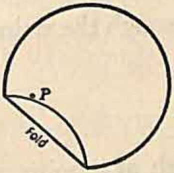


FIG. 39

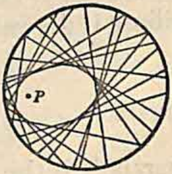


FIG. 40

5. *Plotting on graph paper.* An ellipse is the path of a point which moves so that its distance from a fixed point is always a given fraction of its distance from a fixed line, i.e. in Fig. 41, for all positions of *P*,

$$PF = k \times CP,$$

where *k* is less than 1.

To plot the ellipse, a straight line *AB* is drawn and a point, *F*, marked at some distance from *AB*. In the diagram the value of the fraction used is $\frac{3}{4}$. Points which are 1 cm away from *AB* will lie on the line parallel to *AB*, and 1 cm away from it. The ellipse will cut this line at a distance

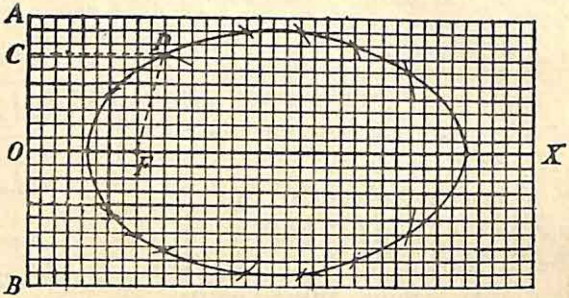


FIG. 41

$\frac{3}{4}$ of 1 cm away from *F*. With centre *F*, radius $\frac{3}{4}$ cm, two arcs are described to cut the 1 cm parallel as shown. A series of parallels are used in the same way. The parallel 2 cm from *AB* is cut by arcs of radius $\frac{3}{4}$ of 2 cm = $1\frac{1}{2}$ cm. The ellipse is drawn through the points so obtained.

## PARABOLA

A parabola is the path of a point which moves so that its distance from a fixed line is equal to its distance from a fixed point. This property gives rise directly to methods 1 and 2 (*a*).

1. A straight line, *CD*, is drawn and a set-square placed against it (Fig. 42). One end of a piece of cotton is fastened to *B*, and the other end to a point *F*. The length of the cotton must be equal to *AB*, the side of the set-square. A pencil is placed at *P* so as to hold the cotton taut against the

side of the set-square. As the set-square is moved along the line $CD$, $P$ traces a parabola.

$$FP + PB = AB = AP + PB$$
$$\therefore FP = AP.$$

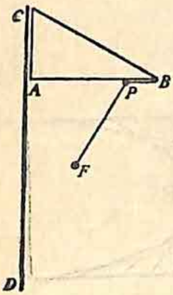


FIG. 42

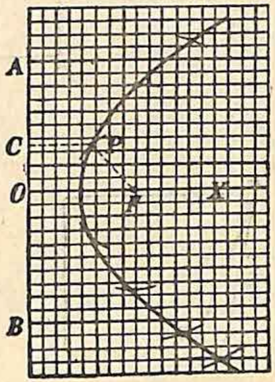


FIG. 43

2. *Plotting on graph paper.* (*a*) This is similar to construction for the ellipse, except that the radii of the arcs which cut the parallels are equal to the distance of the parallels from $AB$.

$$\text{i.e. } FP = CP \text{ (Fig. 43).}$$

The construction is slightly easier for pupils if a series of arcs of radii, $\frac{1}{2}$ cm, 1 cm, $1\frac{1}{2}$ cm, etc., are drawn with centre $F$. On each arc are found the points which lie at a distance from the fixed line equal to the radius of the arc.

(*b*) The parabola is also the curve of $y=x^2$ (see page 92).

3. *Paper folding.* A point $P$ is marked on a sheet of paper near a straight edge, $AB$ (Fig. 44). The paper is folded so that $AB$ passes through $P$ (Fig. 45). This is

repeated several times; the creases form the envelope of a parabola (Fig. 46).

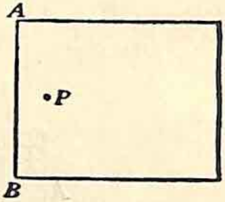


FIG. 44

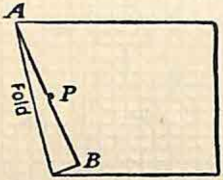


FIG. 45

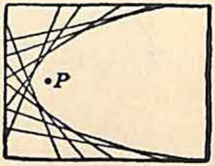


FIG. 46

4. *Envelope.* The envelope of a parabola may also be drawn by the following methods:

(*a*) A straight line $AB$ is drawn, and a point $P$ is marked near it (Fig. 47). A set-square is placed so that the right-angle corner touches $AB$, and one of the sides passes through $P$ (Fig. 48). A line is drawn along the farther arm of the right angle. This is repeated several times with the set-square in different positions to give the envelope of a parabola (Fig. 49).

(*b*) A pair of straight lines $AB$, $AC$, are drawn at any angle to each other (Fig. 50). Points are marked along these lines at regular intervals. A line is drawn from the first division of $AC$ to the last division on $AB$ (Fig. 51). Then moving the ruler from one point to the next along each arm, a series

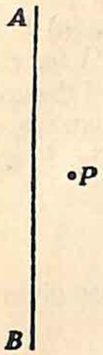


FIG. 47

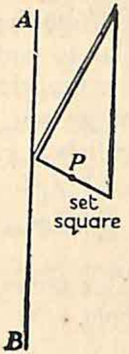


FIG. 48

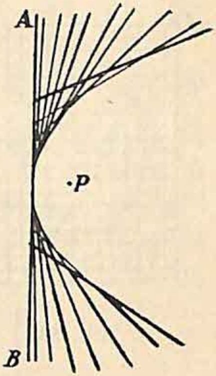


FIG. 49

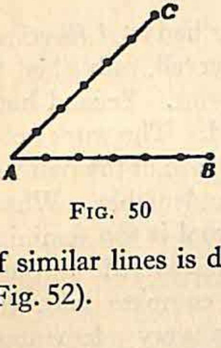


FIG. 50

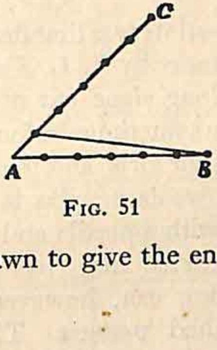


FIG. 51

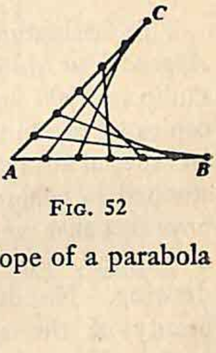


FIG. 52

of similar lines is drawn to give the envelope of a parabola (Fig. 52).

## HYPERBOLA

The hyperbola is the path of a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed line. The ratio must be greater than 1. The following constructions may be used.

1. *Paper folding*. This is similar to the paper-folding method (4) used to obtain the envelope of an ellipse, but in this case the point *P* lies outside the circle. The circle cannot be cut out; tracing-paper should be used.

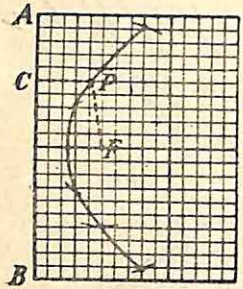


FIG. 53

2. *Plotting on graph paper*. (*a*) This is similar to construction (5) for the ellipse, except that the radii of the arcs are now greater than the distance of the parallels from $AB$. In Fig. 53 the ratio

$$FP{:}PC = 1{\cdot}2{:}1.$$

(*b*) The curve $y = 1/x$ also gives a hyperbola.

## CURVE STITCHING

This fascinating pastime was first described in *A Rhythmic Approach to Mathematics* by E. L. Somervell, published by Philip in 1906 and long since out of print. Second-hand copies are rare but can sometimes be found. The work holds an especial attraction for girls, and the charm of the patterns stitched in coloured wools or silks is undeniable. Where boys feel that work with a needle and wool is too feminine a pastime, the patterns may be constructed through drawing. No drawing can, however, compare with the beauty of the stitched pattern. This very elementary study of envelopes (curves obtained from families of straight lines) can be extremely popular. It is a good example of the fascination that the secondary school pupil can find in a truly mathematical idea.

1. *Parabola*. The work uses certain basic curves, and these are combined in various ways to form patterns. One of the simplest of the basic curves to make is the parabola. Two lines, $AB$ and $AC$, are drawn on a piece of thin card. The angle at which they meet is not important. Points are marked along these lines at $\frac{1}{2}$ cm intervals and pricked

through with a needle or compass point. The holes are numbered very lightly in pencil (Fig. 54).

The numbering given in Fig. 54 will be used as an example. A stitch is made from point 1 to point 13 (Fig. 55). The thread is taken under the card through 13 and brought to the top again through 14. A stitch is made from 14 to 2, and the thread is taken under the card from 2 to 3. Another stitch is made from 3 to 15, and the

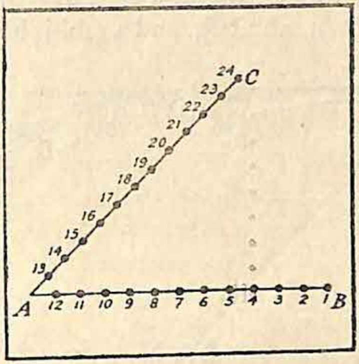


FIG. 54

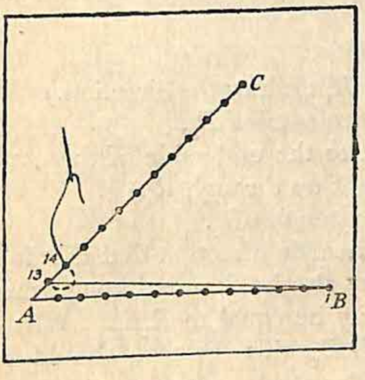


FIG. 55

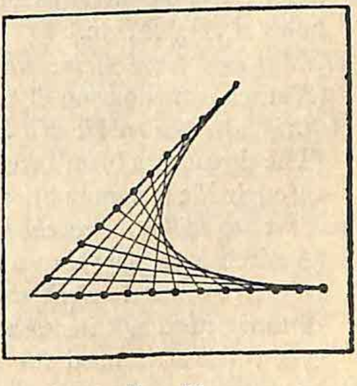


FIG. 56

work continues in this manner. Finally, stitches are made along the lines $AB$, $AC$, to cover the holes, and the pencil figures are rubbed out. The stitches form the envelope of a parabola (Fig. 56).

2. *Curve of pursuit.* This is the path followed by one object when moving towards another object which is travelling in a different direction. It is exemplified clearly by the 'dog-rabbit' situation.

Three points are marked on a card to represent the positions of a dog, $D_1$, a rabbit $R_1$, and a rabbit hole $H$ (Fig. 57).

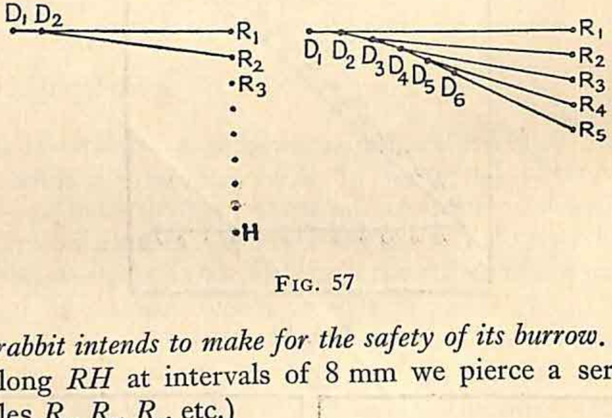


FIG. 57

*The rabbit intends to make for the safety of its burrow.*

(Along $RH$ at intervals of 8 mm we pierce a series of holes $R_1$, $R_2$, $R_3$, etc.)

*The dog runs towards the rabbit.*

(A stitch is made from $D_1$ to $R_1$ to show the direction.)

*A little later the rabbit will have reached* $R_2$,

(The thread has been taken to the underside of the card through $R_1$; it is now brought up through $R_2$.)

*and the dog will have reached a position* $D_2$.

(A stitch $R_2D_2$ now shows the new direction that the dog will have to take. $D_1D_2$ may also be 8 mm although this distance need not necessarily be equal to $R_1R_2$. Whatever interval is used for $D_1D_2$, this distance should be uniform throughout the dog's path.)

*By the time the dog reaches* $D_3$ *in the new direction,*
(The thread has been taken to the underside of the card through $D_2$; it is now brought up through $D_3$ which has been measured along $R_2D_2$.)
*the rabbit will have reached* $R_3$;
(A stitch is made $D_3R_3$.)
*and by the time that the rabbit reaches* $R_4$,
(The thread is brought under the card and up through $R_4$.)
*the dog will have reached a position* $D_4$,
*and so on.*

It will be noticed that the holes showing the path of the rabbit can all be marked out before stitching is started, but those for the path of the dog can only be made as the work proceeds. Each fresh position of the dog is obtained along the last stitch made on the upper side of the card.

In early work the distances in each set should be regular, i.e., $R_1R_2 = R_2R_3 = R_3R_4$, etc., and $D_1D_2 = D_2D_3 = D_3D_4$, etc. It is probably easier if $D_1D_2$ is made equal to $R_1R_2$ at first. Later on the distances in each set can vary, but the variation should follow some rule, e.g. increasing or decreasing by 2 mm each step. Thus $R_1$ $R_2$ would be 8 mm, $R_2$ $R_3$ would be 10 mm, $R_3$ $R_4$ 12 mm etc. By altering the original positions, $R_1$, $D_1$, $H$, a large range of curves is obtained. The path, $R_1$ $R_2$ $R_3$, etc., need not be a straight line: it can be part of a circle, or any other locus, so adding still further variation to the curves obtained.

A pair of dividers or compasses is useful for marking off regular distances along each path.

3. *The tractory.* This can be explained as follows: A man follows the path $M_1M_2M_3$, etc. (Fig. 58). He holds one end of a stick, the other end is at $S_1$ when the man is at $M_1$. A stitch, $S_1M_1$, shows this first position of the

stick. When the man is at $M_2$, the stick takes up the position $M_2S_2$. The position $S_2$ is found by opening dividers or compasses to a distance equal to $M_1S_1$, and marking off from $M_2$ to a point $S_2$ on the stitch $M_1S_1$. The stitch $M_2S_2$ is made, and the process repeated as required. All the distances $M_1S_1$, $M_2S_2$, $M_3S_3$, etc., representing the length of the stick, are equal; the points $S_2$, $S_3$, $S_4$, etc., all lie on the last stitch made.

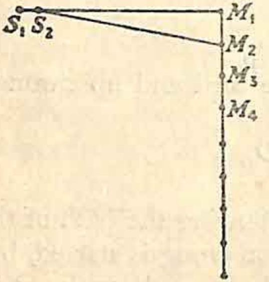


FIG. 58

Some interesting results can be obtained from this curve by varying the starting position of the stick, also by varying the path followed by the man, e.g. a circle, etc.

4. *The circle.* Many interesting patterns can be based on circles. By one method, concentric circles are drawn and holes are pricked around each circumference. Stitches are made from one circle to the other. The resulting patterns vary according to the radii of the circles and the number of points on each circumference. For any one pair of circles, the numbers of points on the circumferences should be in some simple ratio, e.g. twice as many on the outer circle as on the inner circle. The stitches are continued until they begin to repeat themselves (i.e. another stitch made between the same two holes).

For another series of patterns two, three, or four overlapping circles can be used, stitches joining holes on one circle to those on another. The circles should be of equal radii, with their centres placed in regular order to form an equilateral triangle or square.

It is advisable to give only one or two basic curves at first, and allow the children to use these to form their own patterns. The parabola is the simplest, and can form the basis of many attractive and easy patterns. Some patterns of simple construction should be shown to the class so that they can see how patterns are evolved. Their first attempts should not be allowed to become too ambitious. Much time and temper will be saved if the pattern to be made is first roughly sketched in pencil with freehand lines showing the stitches. For satisfying results all necessary measurements must be accurate; guesswork should not be permitted. As skill is acquired, other curves can be introduced, and still more will no doubt be invented.

## XV

## AVERAGES: STATISTICS

EVERY class offers ample material for work on averages with the opportunity for testing and using other processes as well. The interest and enthusiasm aroused in the pupils when they find themselves used as a source of mathematics far outweighs the advantage of the graded examples of the textbook.

For a first demonstration of average probably the best example to choose is height. This average is visual in a way in which age, weight, etc., are not. The class can be asked to decide who is the tallest pupil and who is the shortest. They stand against the blackboard and a mark is made to indicate each height. These heights show the range, the other pupils' heights lie between them. Some height between the two extremes gives an indication of the normal height of the children in the class.

The class is then divided into groups of about 6-8 pupils. The groups should not be selected in any manner likely to influence averages; an arrangement by age must not be used. The members of each group measure each other's heights to the nearest centimetre. Every child keeps a record of the heights in his group. Paper scales fastened to the class-room walls at convenient points enable the measuring to be done quickly and easily. The scales need not be calibrated in centimetres throughout their lengths. For the first part, metres only need be shown, with centimetres marked between 1·20 metres and 1·60 metres. The actual range needed depends on the class.

The heights are arranged in order of size, and a written description made of the record of each group. A block diagram can show the height of each member of the group (Fig. 1). The total heights of the groups can be found and compared. The comparison is only fair if the groups are of equal size. On the other hand, a comparison of unequal groups leads easily to the notion of average, if the total height of each group is shared between the members of the group. A line is drawn on the diagram at the average height. The children should see that the amounts above the average correspond to the amounts below the average.

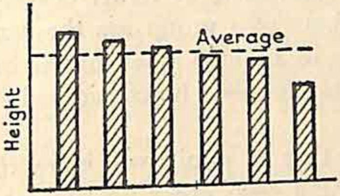


FIG. 1

The average of the whole class can then be found. There is no need to add together each separate height, since group totals have already been found. They are added together, and the class total is divided by the number of pupils. This calculation will be the same for all members of the class. If the groups are of equal size, the group averages can be added, and then divided by the number of groups, to give the same result. This cannot be done if the groups are of unequal sizes.

Averages of ages, weights, distances from home to school (measured in paces), number in family, etc., can be treated in a similar manner. The work should always be done through groups. Class organization is made simpler, and the less able children can follow the work more readily when it deals with the small number contained in a group.

Ages can be reckoned in terms of years and months. When all members of a group are the same age in years, and differ only in months, they might be guided to see that only the months need be considered in order to find the average.

It is unlikely that all pupils will know their weight. In any case, weighing each other on the school scales is an excellent practical exercise, and all children should have the opportunity to carry it out. Only one group at a time will be able to weigh, since only one set of scales is likely to be available. The other groups can be engaged in recording other results.

For ages, weights, and heights, the average found can be a real one. It is possible, of course, that one or two pupils in the class may be the average height. When the size of families is considered, the average is likely to be a value such as 5·8, which is impossible in terms of real families. An intelligent class might see the difference between the two types of quantities under discussion. Height is a continuous quantity; as a child grows from 1·20 m to 1·60 m in height, he is, at some time or another, every conceivable height between these two limits. Other quantities increase in a series of 'jumps'. The number in a class may increase from 35 to 36. The increase is made in one step; there is not a continuous increase from 35 to 36 through a range of intermediate values. Nevertheless, averages present us with just such values. The average of classes of 35 and 36 pupils is 35·5. This is a true representation of the situation, yet it is an impossible value in reality.

## STATISTICS

The term *statistics* is used in two ways. It may denote a collection of numerical facts, or it may mean the mathematical treatment of these numerical collections. Statistics of the first kind will certainly be studied in most schools. Much of the work needed for statistical calculations lies beyond the scope of the school, although some of the ideas of statistics can be shown and used in simple cases.

On the whole the treatment of statistics should be incidental to other work, taking opportunities to use statistical methods, or point out statistical ideas, as occasion arises. Some time may be given to studies which emphasize particular statistical techniques. The first, and probably the most important, step in the treatment of statistics is to encourage the pupils to look at numerical data intelligently. The facts studied should, as far as possible, relate to the children, or be the results of their own observations. The subject then takes on a vitality which cannot be found in sets of figures taken solely from reference books.

## A CAR CENSUS

For example, a large car manufacturer may state in advertisements: '*One in every four cars on the road is a particular make*'. Observation by a class throughout a week or a fortnight can check this statement. The record is easily kept by the 'gate' method (Fig. 2; see also page 42). The observations are not made in school time, but any other time when the children are free and feel so

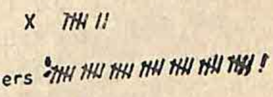


FIG. 2

disposed. Each child keeps his own record, observing cars passing on any road. The records are collected and totalled daily.

An idea of sampling emerges from this simple study. Discussion can give emphasis to the points made. We wish to know something about the ratio of the number of a particular make to other cars on the roads. To find the exact truth is an impossible task. A small sample taken on a quiet road over a space of five or ten minutes is obviously not reliable. A large sample, such as can be collected by a class over a space of a week, gives a much clearer indication. The reliability of the 'sample' may be checked by comparing results for two separate weeks, or by dividing the class into two groups, and comparing their findings. Results will not be identical, but will show a very large measure of agreement. It is reasonable to suppose that yet another sample would still lead to a similar result. This point is considerably strengthened if a third sample is taken and found not to differ appreciably from the others.

The importance of the size of the sample should be brought out in some way. From the work already done, we may decide that *X*'s form about 25 per cent of the car population. A set of samples of ten cars each will show wide deviations from this, so demonstrating the unreliability of small samples. Samples of 20, 50, 100, 500 cars will probably show progressively decreasing deviations. Exact agreement will not be obtained even with very large samples, indicating that final results must not be stated to too high a degree of accuracy. The final calculation may give an answer of 21·27 per cent; this should be approximated to 21 per cent or even 20 per cent.

The minimum size for a satisfactory sample might be briefly discussed. Its determination is not an easy matter, but the fact that it exists is not difficult to grasp.

The validity of the results obtained may be discussed. The sample is a local one. It is likely with a large number of observers in a small area that some cars have been recorded many times. Any census taken near a car factory will be biased, and not a normal sample.

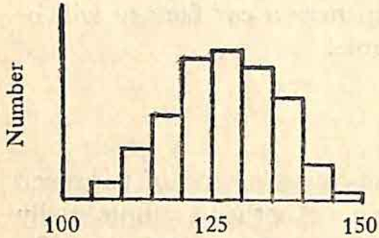
## SAMPLING

We have used the 'Car Census' as a means of approach to the statistical idea of sampling. Another example easily treated in the school is the letter frequency count (page 143).

## DISTRIBUTION

The method of presenting any collected information is important and should be discussed. Various graphical methods are possible and tabulation can be used in different ways. It is necessary to find a form which shows the salient points clearly, but without distorting them in any way. Skill in presentation is acquired slowly, and long, careful training is needed. In most of the work the graphical forms will be found invaluable in bringing trends to the pupils' notice, or showing cause and effect. Graphs of heights and ages, weights and ages, heights and weights, can be constructed from data available in the school. Girls may be interested in the weekly records of babies' weights.

In many cases, frequency distribution graphs can be made from collected data. Fig. 3 shows a frequency distribution of heights of children. The height of each column indicates the number of children whose heights lie within a particular range, e.g. 105-110 cm, 110-115 cm, etc. When the numbers used are large enough, most graphs of this type show a marked resemblance to one another, a central peak, falling away rapidly on either side. With very large samples the curve of normal distribution is obtained (Fig. 4).

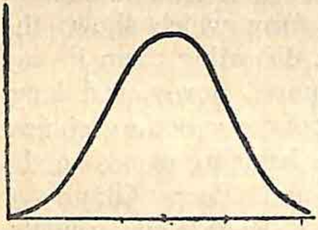


Heights in cm

FIG. 3

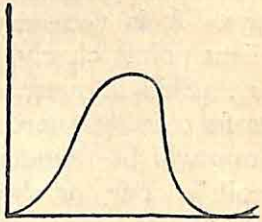
Not all curves of very large samples are curves of normal distribution. That of the Intelligence Quotients of students who have failed examinations, is skew; the upper range of I.Q.s of the whole population is not likely to be found among the failures (Fig. 5).

An indication of the skewness of a curve is given by the difference between the *mean* and the *median* of the observations. The mean is the ordinary arithmetical average; the median is the central observation, i.e., there are as many observations above it as below it. In a curve of normal distribution the mean and the median are the same.



Normal distribution

FIG. 4



I.Q. of students who have failed examinations

FIG. 5

## SCATTER

It will be realized that the average of a set of results does not give a complete picture by itself. Two sets of examination marks may have the same average, 52, but the range of marks in one case may be 0-93, and in the other case 41-64. Some knowledge is required of the spread or scatter of the results. The range, i.e. the difference between the highest and lowest observations, is not satisfactory, as it uses individual extreme cases.

The *inter-quartile range* is more satisfactory and is easily found. The results are arranged in order and the median found, so dividing the observations into upper and lower halves. The median of each half is found, thus dividing the observations into four equal groups. The upper median is known as the upper quartile the lower one as the lower quartile. The difference between them is the inter-quartile range.

The measure of scatter most commonly used is the *standard deviation*. It is rather more difficult to calculate, but its significance remains the same. It is a measure of the way in which the separate items of a set of observations deviate from the average.

## CORRELATION

Some idea of *correlation*, the association between two varying quantities, can also be given. There are many cases where a positive correlation is expected from common observation, e.g. heights and ages. The relationship is not always clear, however, and statistical methods are employed to establish the degree of correlation. The nature of correlation can be shown by a simple graphical method. The two quantities, say, height and weight, are plotted on two axes (Fig. 6). Each observation of the height and weight of a person is represented by a point. Thus $A$ represents a person 1·8 m in height and weighing 72 kg. $B$ represents a person 1·35 m in height and weighing 36 kg. All the

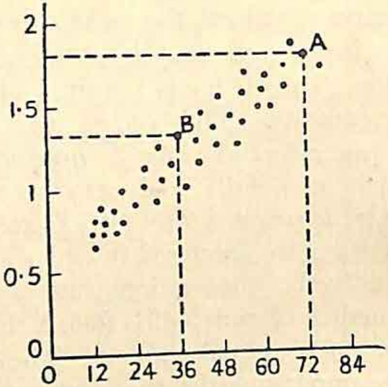
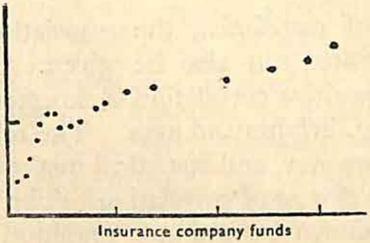


FIG. 6

other points are plotted in an identical manner. When some definite relationship exists, these points bunch together around a line (known as the *line of regression*). Where no correlation exists, the points are spread indiscriminately over the graph paper.

The fact that a high degree of correlation exists between two quantities does not necessarily mean that one is directly affected by the other. Fig. 7 would appear to show that



Each point shows the amount of insurance company funds and the number of mental defectives for one year

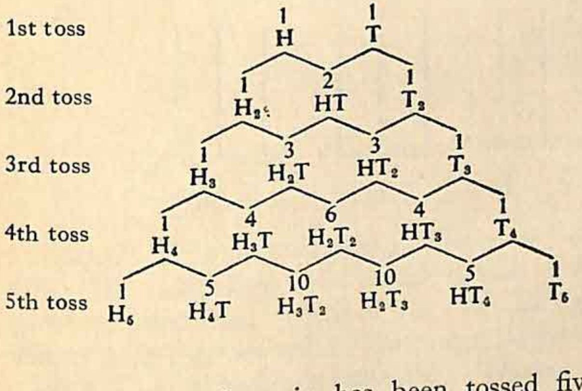
FIG. 7

there is some connexion between the number of mental defectives and the size of insurance company funds. Both are outcomes of a growing community; there is no other relationship.

### PROBABILITY

The ideas of probability can be demonstrated by coin-tossing. If a rupee coin is tossed the result will be either head or tail; each has an equal chance of occurring. If the coin is tossed again the possible sequences are now HH, HT, TH, TT (HH indicates two successive heads). The chance of obtaining two heads following, or two tails following, is one out of four. The chance of one of each (HT or TH) is two out of four.

The argument can be repeated for as many tosses as one likes, and the chance of any particular combination of heads and tails can be found. The work is best done using the number structure known as Pascal's triangle.



($H_2T$ indicates a combination of two heads and one tail.)

and so on.

By the time the coin has been tossed five times the number of possible combinations is

$$1+5+10+10+5+1=32.$$

The chance of five successive heads is 1 out of 32. The chance of four heads and one tail is 5 out of 32. The sequences which give four heads and one tail are HHHHT, HHHTH, HHTHH, HTHHH, THHHH (HHTHH indicates two heads followed by a tail, followed by two heads).

The above argument is not confined to coin tossing of course. It can be applied to all cases where chances are even, e.g. when a die is thrown. The method can be extended without much difficulty to instances such as selecting one object from three, etc.

Statistical methods are also used when dealing with very large numbers, e.g. the forecasting of population figures (pp. 148-50). The premiums of insurance companies depend on the normal expectation of life. While this cannot be stated for any one individual, it can be stated for a large population with a high degree of precision.

# XVI

# HISTORICAL CALCULATING METHODS

SOME of the calculating methods used in the past, or in other parts of the world, are sure to interest school pupils. The able pupils will often achieve a stronger sense of number structures from their study; for pupils of low ability the abacus, in particular, may provide valuable remedial treatment.

## THE ABACUS

Some examples of abacal calculations are given below:

(*a*) *Add 373 + 256*

*Step 1*. The number 373 is placed on the abacus (Fig. 1).

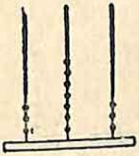


FIG. 1

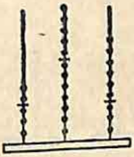


FIG. 2

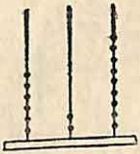


FIG. 3

*Step 2*. The number 256 is also placed on the abacus (Fig. 2). Less able children are helped at this stage if a piece of cardboard is placed on each rod after the first number, so that the two numbers can be seen separately.

*Step 3*. The rods are inspected. The tens rod contains 12 beads; 10 are taken off and replaced by 1 bead on the hundreds rod.

*Step 4*. The answer, 629, is read off the abacus (Fig. 3).

(*b*) *Subtract 235 from 876*

*Step 1.* The number 876 is placed on the abacus (Fig. 4).
*Step 2.* The 5 units are taken from the units rod (Fig. 5).
*Step 3.* The 3 tens are taken from the tens rod (Fig. 6).
*Step 4.* The 2 hundreds are taken from the hundreds rod (Fig. 7).
*Step 5.* The answer, 641, is read off the abacus.

The above example does not involve adjustment. As will be seen from the following example, the adjustment is carried out through the 'decomposition' method.

*Subtract 269 from 876*

*Step 1.* The number 876 is placed on the abacus (Fig. 8).
*Step 2.* We cannot take nine beads from the units rod. A bead is taken from the tens rod and ten beads added to the units rod (Fig. 9).
*Step 3.* Nine beads are taken from the units rod (Fig. 10).
*Step 4.* Six beads are taken from the tens rod (Fig. 11).

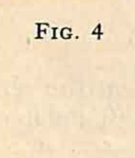


FIG. 4

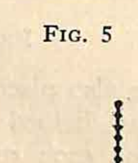


FIG. 5

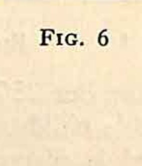


FIG. 6

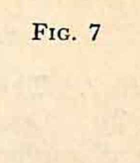


FIG. 7

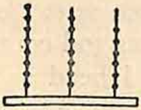


FIG. 8

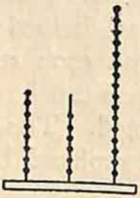


FIG. 9

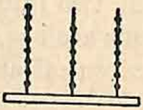


FIG. 10

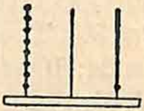


FIG. 11

*Step 5.* Two beads are taken from the hundreds rod (Fig. 12).

*Step 6.* The answer, 607, is read off the abacus.

(*c*) *Multiplication.* To understand the method of multiplying we must first realize that if a set of beads are moved one rod to the left, the number they represent is multiplied by 10. Thus, 23 becomes 230 (Figs. 13, 14). The method of multiplication is a combination of this multiplication by 10, 100, etc., and of continued addition.

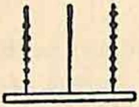


FIG. 12

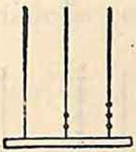


FIG. 13

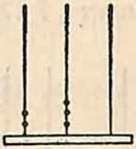


FIG. 14

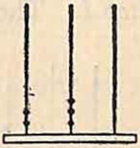


FIG. 15

*Multiply 23 by 14*

*Step 1.* 23 × 10. 23 is set on the abacus and then moved one rod to the left to give 230 (Fig. 15).

*Step 2.* 23 × 4. 23 is placed on the abacus four times (Fig. 16).

*Step 3.* The rods are inspected and any necessary 'carrying' is done.

*Step 4.* The answer, 322, is read off the abacus (Fig. 17).

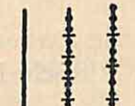


FIG. 16

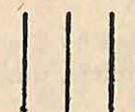


FIG. 17

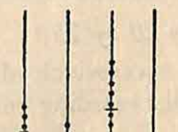


FIG. 18

*Multiply 325 by 23*

*Step 1.* 325×20. 325×10 is set on the abacus twice (Fig. 18). The use of a large number of beads can be avoided by 'carrying' as the work proceeds (Fig. 19).

*Step 2.* 325×3. 325 is set on the abacus three times (Fig. 20).

*Step 3.* Carrying is completed.

*Step 4.* The answer, 7,475 is read off the abacus (Fig. 21).

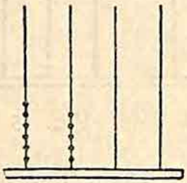


FIG. 19

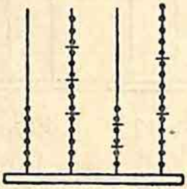


FIG. 20

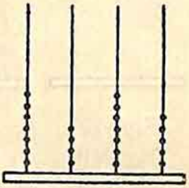


FIG. 21

Division was a difficult process on the abacus and needed at least as much mental agility as the modern method. Few people could do it.

## DUPLICATION

The method of multiplying by duplication is still used in some parts of the world. It was seen in use in a Czech evacuee school in London during World War II.

*Multiply 20 by 25*

25 is successively divided by 2 and the answers written down, disregarding remainders, until 1 is reached

25 12 6 3 1

20 is multiplied by 2 a corresponding number of times, and the answers written below the first series.

| | | | | |
|---|---|---|---|---|
| 25 | 12 | 6 | 3 | 1 |
| 20 | 40 | 80 | 160 | 320 |
| (20×1) | (20×2) | (20×4) | (20×8) | (20×16) |

Numbers in the lower line which lie below an even number in the top line are crossed off.

| | | | | |
|---|---|---|---|---|
| 25 | 12 | 6 | 3 | 1 |
| 20 | ~~40~~ | ~~80~~ | 160 | 320 |

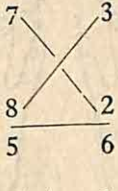
The other numbers in the bottom line are added to give the required answer.

$$20 + 160 + 320 = 500.$$

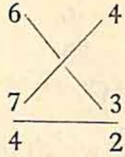
[The answer is obtained by adding (20×1) + (20×8) + (20×16), which is equivalent to (20×25).]

## THE MULTIPLICATION SYMBOL

The sign for multiplication, ×, is said to come from a method in common use a few hundred years ago. Only the multiplication tables as far as the five-times table were needed. To multiply 7 by 8 the two numbers are written



one below the other. Each is subtracted from 10, and the results written by the side. These two numbers are multiplied together to give the units figure of the answer (3×2=6). The tens figure is obtained by subtracting either 2 from 7, or 3 from 8.



In multiplying 6 by 7 by this method, a ten is carried from the units figure.

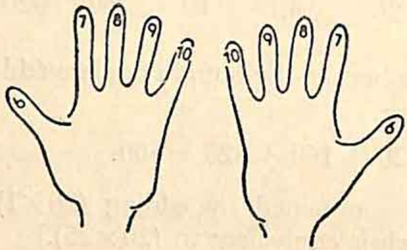


FIG. 22

Finger reckoning uses the same basic principles as the previous method. The fingers on each hand are numbered from 6 to 10 starting with the thumb (Fig. 22). To multiply 8 by 9, finger 8 on one hand is touched by finger 9 on the other hand (Fig. 23). The tens figure of the answer is

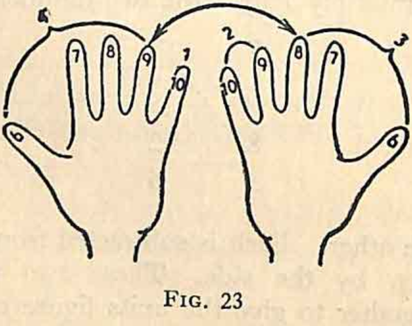


FIG. 23

obtained by counting the number of fingers on both hands up to and including the touching fingers (4+3=7). The units figure is found by multiplying together the numbers of fingers beyond the touching fingers (2×1 = 2).

It is not suggested that these methods should supplant a knowledge of the tables, but most children find them of great interest.

## NAPIER'S RODS

Napier's rods, also known as Napier's bones, were first described by Lord John Napier in 1617. They became a popular device for carrying out long multiplication. They consist of a set of strips, on each of which is written one of the multiplication tables from 0 to 9 (Fig. 24). The diagonal lines across each square of the strips divide the products into tens and units; these lines should be heavily marked. A frame in which to place the strips (Fig. 25) is useful, but not essential.

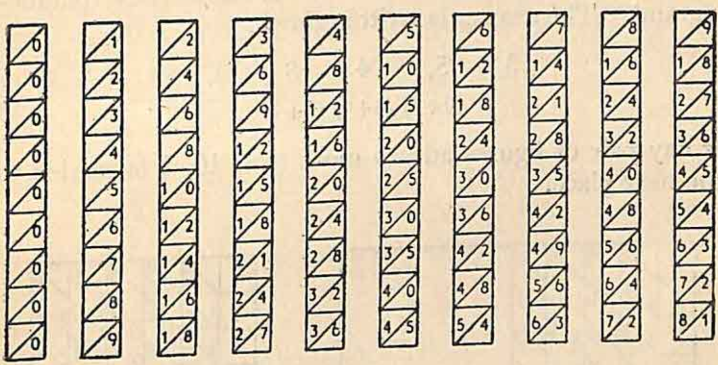


FIG. 24

To multiply 5069 by 7, the 5, 0, 6, and 9 strips are placed in the frame as shown in Fig. 26, so forming the number 5069. Along row 7 will be the products, 7×9, 7×6, etc.

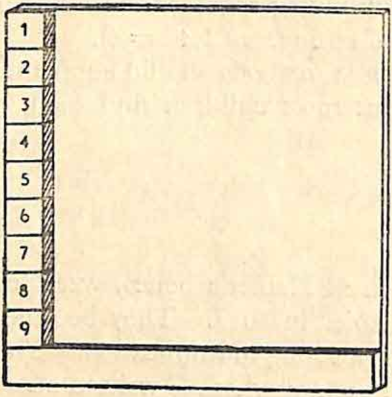


FIG. 25

The answer is read off, starting with the units figure. The first figure, 3, is the units figure of the answer. The 6 tens of the 63, and the 2 tens of the 420 ($60\times 7$) are added together to give the tens figure of the answer ($6+2=8$). The figure for each place in the answer is similarly found by adding the figures in each 'parallelogram'. The answer is written down,

$$\begin{matrix} 3 & 5, & 4 & 8 & 3. \\ & (0+5) & (4+0) & (6+2) & \end{matrix}$$

If any pair of figures add to more than 10, 1 is carried to the next place.

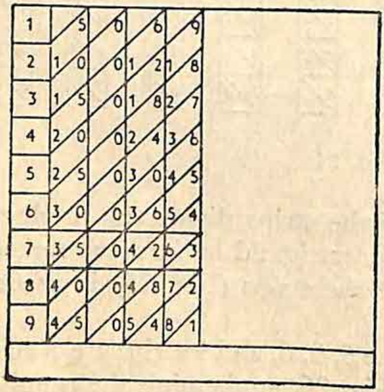


FIG. 26

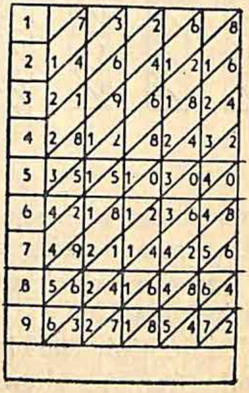


FIG. 27

For long multiplication, the partial products are written down using the rods, and then added.

*73,268 × 58* (Fig. 27)

$$\begin{aligned} 73{,}268 \times 8 &= 586{,}144 \\ 73{,}268 \times 5 \times 10 &= \overline{3{,}663{,}400} \\ 73{,}268 \times 58 &= \overline{4{,}249{,}544} \end{aligned}$$

With a little practice quite lengthy multiplication examples can be done rapidly. As in logarithms, which were also invented by Napier, only the process of addition is needed in order to multiply.

# XVII

# DIVERSIONS

## MAGIC SQUARES

A MAGIC square is a set of numbers arranged to form a square, so that the sum of the numbers in every column, row, and diagonal is the same. They have been in existence for some thousands of years, and were used as magic charms in Europe in the Middle Ages. (One is shown in an engraving by Dürer.) They are still so used in Africa, India, and other parts of the world.

Children can be given uncompleted squares and asked to insert the missing numbers. A large amount of simple practice in addition and subtraction may be obtained in this way. The use of the diagonals will probably need emphasis for it is easy to fall into the habit of using only the vertical and horizontal rows.

To compile an original magic square is too difficult for most children but given a magic square, others can easily be formed from it. These are known as transformations. Geometrical transformations are those in which the numbers

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

FIG. 1

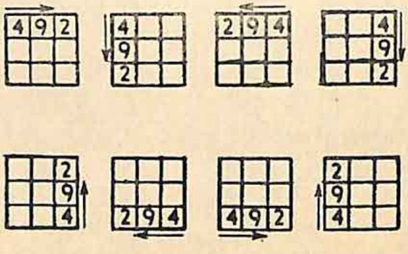
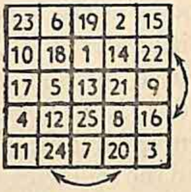


FIG. 2

remain unaltered, but the positions of the rows and columns are changed. In the simple square shown in Fig. 1, the 4 may occupy any one of the four corner positions, and the 4, 9, 2 row occupy any one of eight positions as shown in Fig. 2. The other rows fall into place with it, of course.

A magic square may also be transformed by interchanging rows which are equidistant from the centre of the square; the corresponding columns must also be interchanged (Fig. 3).



| | | | | |
|---|---|---|---|---|
| 23 | 6 | 19 | 2 | 15 |
| 10 | 18 | 1 | 14 | 22 |
| 17 | 5 | 13 | 21 | 9 |
| 4 | 12 | 25 | 8 | 16 |
| 11 | 24 | 7 | 20 | 3 |

FIG. 3

Transformations may also be made by:

(i) Adding the same number to each of the numbers in a magic square.

(ii) Subtracting each of the numbers in a magic square from a given number.

(iii) Multiplying each of the numbers in a magic square by a given number.

(iv) Dividing each of the numbers in a magic square by a given number.

The latter method will lead to a magic square with fractional numbers. For example, if each of the numbers of Fig. 1 is divided by 12, the top row (4, 9, 2) becomes $\frac{1}{3}$, $\frac{3}{4}$, $\frac{1}{6}$. This form of magic square can give practice in addition and subtraction of fractions.

Practice in money and measures may also be provided by magic squares if each of the numbers in a magic square indicates a number of units of money, length, etc. Thus, the numbers of Fig. 1 may refer to 10 paise, in which case the top row becomes 40p., 90p., 20p., the total being Re 1·50.

The magic square shown in Fig. 1 is the only basic one which can be constructed from nine numbers; all other nine-number squares are transformations from it.

The number of rows or columns in a magic square is known as the order of the square. Fig. 1 is a square of the third order; Fig. 3 is a square of the fifth order. There is a simple method of constructing magic squares of an odd order (3, 5, 7, etc.). Start by inserting 1 at the top of the centre column; 2 is inserted at the foot of the next column, and the following numbers in order along a diagonal. When a diagonal reaches the side of the square it is continued on the other side (Fig. 4). When we come to a cell already used, the next number is placed below the last cell filled in and the diagonal continued from there. The complete square is shown in Fig. 5.

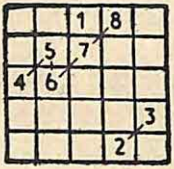


FIG. 4

| | | | | |
|---|---|---|---|---|
| 17 | 24 | 1 | 8 | 15 |
| 23 | 5 | 7 | 14 | 16 |
| 4 | 6 | 13 | 20 | 22 |
| 10 | 12 | 19 | 21 | 3 |
| 11 | 18 | 25 | 2 | 9 |

FIG. 5

There is also a simple method of constructing magic squares of doubly-even order, 4, 8, etc. The numbers are first written in order in the rows of the square (Fig. 6).

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

FIG. 6

| | | | |
|---|---|---|---|
| 16 | 2 | 3 | 13 |
| 5 | 11 | 10 | 8 |
| 9 | 7 | 6 | 12 |
| 4 | 14 | 15 | 1 |

FIG. 7

Cells along the diagonals are then interchanged across the centre of the square (Fig. 7).

A very comprehensive treatment of magic squares is given in *Mathematical Recreations* by M. Kraitchik (George Allen and Unwin Ltd.).

## NUMBER PATTERNS

At all times children should be encouraged to look for pattern in mathematics, for it is through realizing the pattern of behaviour that many of the essential mathematical structures are seen and understood. Number patterns form a good starting-point. Children undoubtedly obtain pleasure from discovering examples of patterned behaviour in numbers, and this pleasure can be the beginning of intellectual delight in the subject. For example, few people discover the pattern of the sum of the odd numbers, shown below, without experiencing considerable mental satisfaction in the simplicity and orderliness of the number behaviour.

$$\begin{aligned} 1 &= 1 = 1\times1 \\ 1+3 &= 4 = 2\times2 \\ 1+3+5 &= 9 = 3\times3 \\ 1+3+5+7 &= 16 = 4\times4, \text{ etc.} \end{aligned}$$

Other examples of number patterns are:

(*a*)

$$\begin{aligned} 37\times3 &= 111 \\ 37\times6 &= 222 \\ 37\times9 &= 333 \\ 37\times12 &= 444, \text{ etc.} \end{aligned}$$

(*b*)

$$\begin{aligned} 1\times1 &= 1 \\ 11\times11 &= 121 \\ 111\times111 &= 12321 \\ 1111\times1111 &= 1234321, \text{ etc.} \end{aligned}$$

(*c*)

$$\begin{aligned}(1\times10)+2 &= 12\\(12\times10)+3 &= 123\\(123\times10)+4 &= 1234\\(1234\times10)+5 &= 12345, \text{ etc.}\end{aligned}$$

(*d*)

$$\begin{aligned}(1\times9)+2 &= 11\\(12\times9)+3 &= 111\\(123\times9)+4 &= 1111\\(1234\times9)+5 &= 11111, \text{ etc.}\end{aligned}$$

(*e*)

$$\begin{aligned}(1\times8)+1 &= 9\\(12\times8)+2 &= 98\\(123\times8)+3 &= 987\\(1234\times8)+4 &= 9876, \text{ etc.}\end{aligned}$$

(*f*)

$$\begin{aligned}(1\times8)+1 &= 9\\(11\times8)+11 &= 99\\(111\times8)+111 &= 999\\(1111\times8)+1111 &= 9999, \text{ etc.}\end{aligned}$$

(*g*)

$$\begin{aligned}(1\times11)+2 &= 13\\(12\times11)+3 &= 135\\(123\times11)+4 &= 1357\\(1234\times11)+5 &= 13579\end{aligned}$$

(*h*)

$$\begin{aligned}(9\times9)+7 &= 88\\(98\times9)+6 &= 888\\(987\times9)+5 &= 8888\\(9876\times9)+4 &= 88888, \text{ etc.}\end{aligned}$$

(*i*)

$$\begin{aligned}11\times9 &= 99\\22\times9 &= 198\\33\times9 &= 297\\44\times9 &= 396, \text{ etc.}\end{aligned}$$

(*k*)

$$
\begin{aligned}
99\times 2 &= 198\\
99\times 3 &= 297\\
99\times 4 &= 396\\
99\times 5 &= 495, \text{ etc.}
\end{aligned}
$$

(*l*)

$$
\begin{aligned}
12{,}345{,}679\times 2\times 9 &= 222{,}222{,}222\\
12{,}345{,}679\times 3\times 9 &= 333{,}333{,}333\\
12{,}345{,}679\times 4\times 9 &= 444{,}444{,}444, \text{ etc.}
\end{aligned}
$$

The pupils should be given the left-hand sides only of the above sets. After working out the first few lines of each table, they should be able to write the answers which follow without further calculation. Two or three more lines should then be worked to serve as a check. In some cases, five or six lines only of the table are given, and the pupils may be asked to continue writing the table.

Number series can also be used at quite low levels without any reference to arithmetic progressions, etc. Some can be shown in visual form, e.g. the triangular numbers (Fig. 8).

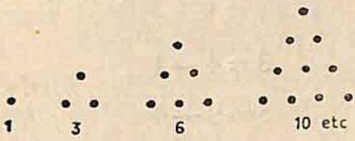


FIG. 8

Inspection of the pattern soon shows the series as 1, 1+2, 1+2+3, 1+2+3+4, etc. Inspection of the numbers show the differences, 2, 3, 4, etc. Another simple set is the square numbers (Fig. 9). If the differences in consecutive patterns are noted (see Fig. 10), the series is seen as 1, 1+3, 1+3+5, 1+3+5+7, etc.

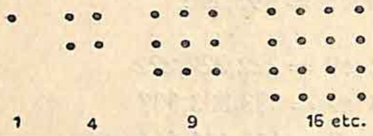


FIG. 9

Having seen the formation of the first few patterns and found the numbers they represent, the children can attempt to decide what the next number will be. Inspection of the numbers of a series, or of the differences between consecutive numbers, generally supplies the necessary clues to continue the series. In the case of geometrical numbers the guesses can be checked by forming the next patterns. Other geometrical numbers can be based on the pentagon, hexagon, etc. The Greeks made a particular study of these numbers, and it is probable that some of their great knowledge of number structures came from observing the patterns. For example, any odd number can be expressed as the difference of two squares:

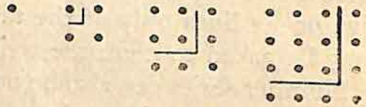


FIG. 10

$$3 = 4 - 1$$

$$5 = 9 - 4$$

$$7 = 16 - 9, \text{ etc.}$$

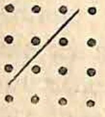


FIG. 11

This property is soon seen from the patterns of square numbers. Similarly, the sum of any two successive triangular numbers is a square number; this property is easily shown by arranging the triangular numbers to form a square (Fig. 11).

## NUMBER PUZZLES

These may either show curious number behaviour, or be of the type in which missing figures have to be inserted.

e.g. $$\begin{array}{rrr} 1 & 3 & 5 \\ +3 & 1 & * \\ \hline * & * & 6 \end{array}$$

Examples of this kind offer greater difficulty than straightforward examples in the four rules. Their advantage is that, in tackling them, the child becomes more aware of the processes of normal calculations. Some help may be needed at first; simple questions of the type $3+? = 8$ form a suitable introduction for children who find difficulty. Children should be encouraged to make up further examples and test each other. The reversal of processes necessary in this work is an important idea which needs to be stressed during the child's first years at school.

Some examples of curious number behaviour are:

(*a*) The number 123,456,789 is formed from the digits written in order. When multiplied by 2, 4, 5, 7, or 8, the answer also contains all the digits, but in a different order. Notice also that 987654321—123456789 is 864197532. Here all the three numbers contain all the nine digits.

(*b*) Write down a number of three figures, e.g. 672

$$\begin{array}{rr} \text{Reverse the figures} & 276 \\ \cline{2-2} \text{Find the difference} & 396 \\ \text{Reverse the figures} & 693 \\ \cline{2-2} \text{Add} & 1089 \\ \cline{2-2} \end{array}$$

Whatever number is first chosen, the result is always 1089.

(*c*) If the operations of (*b*) are carried out with a number of two different figures, the result is always 99. Each step in the working in this version of the puzzle should contain two figures, and where the difference between the first and second lines appears as 9, it should be written 09.

(*d*) If a number of two figures is reversed, the sum of the two numbers so formed is always divisible by 11. Their difference is divisible by 9.

(*e*) Write any number, say, 218.
Make up another number from the same digits, 182.
Their difference is always divisible by 9.

(*f*) Write a number of three figures, with the figures decreasing by ones from left to right, e.g. 987.
Reverse the figures: 789.
Find the difference: $987 - 789 = 198$.
The result is always 198.

(*g*) Write a number of two figures, 28.
Reverse the figures 82.
Find the difference, $82 - 28 = 54$.
The sum of the digits of the answer is always 9.

(*h*) If the operations of (*g*) are repeated with a number of three figures, the result is always of the form $x9y$, where $x+y=9$ (e.g. $681 - 186 = 495$).

(*i*) If 142,857 is multiplied by 2, 3, 4, 5, or 6, the answer contains the same digits in the same order, except that each answer starts at a different digit (e.g. $142{,}857 \times 2 = 285{,}714$).

(*k*) If the sum of the digits of a number can be divided by 3, the number itself can be divided by 3.

(*l*) If the sum of the digits of a number can be divided by 9, the number itself can be divided by 9.

(*m*) The square of a number ending in 5 may be found in the following way:
To find $65^2$.

To the number of tens (6), add one ($6+1=7$).

Multiply these together ($6\times7=42$) to give the number of hundreds, i.e. 4,200).

$$\text{Add } 25 : 4{,}200+25=4{,}225$$
$$65^2=4{,}225.$$

## NUMBER CROSSWORDS

Solving number crosswords (see example below) always proves popular, and a useful range of practice can be obtained. Even greater benefit is obtained if the children make up their own crosswords; the making of clues gives useful practice in expression. It is generally necessary to give the children the patterns into which they fit the crosswords. Graph-paper divided into 1 cm squares is useful for this work.

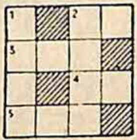


FIG. 12

Across 2. $170\div10$.
3. $183\times2$.
4. Half a century + half a dozen.
5. $75\times4$.

Down 1. $3{,}033+300$.
2. $50+600+1{,}000$.

## TANGRAMS

This is an old Chinese pastime. A square of paper or card is divided as shown in Fig. 13. The pieces are then rearranged to form 'pictures'. A surprising amount of knowledge of simple shapes can be acquired. It is soon discovered that two of the triangles can be arranged to form a square and that a rearrangement of them gives a rhombus. The child may not know the names of all the shapes he uses, but this first knowledge of them proves most valuable when they are later introduced more formally. It is not suggested that much lesson time be devoted to tangrams; they are an occupation for a spare five or ten minutes, or something which the children may well do at home.

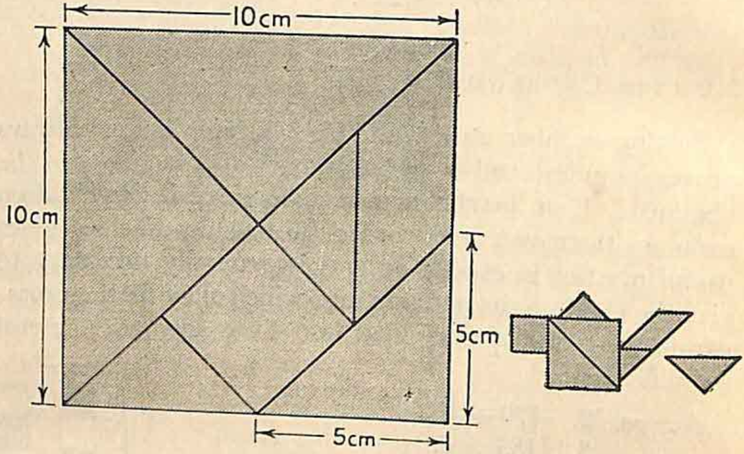


FIG. 13

It is convenient to have a set of card shapes for arranging; these are more easily handled than paper. When a suitable arrangement is made, it can be copied in coloured gummed paper.

## THREE-DIMENSIONAL NOUGHTS AND CROSSES

The game of noughts and crosses is a familiar one. When played in three dimensions it provides excellent practice in solid visualization. For this game we enclose the normal framework within a square (Fig. 14). We now imagine the game played, not on a flat square divided into nine cells, but in a cube divided into three layers of nine cells each (Fig. 15). The noughts and crosses are to be placed in the cells of the cube. The cube is difficult to draw and to use. A simple alternative is to draw three squares to

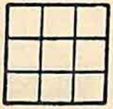


FIG. 14

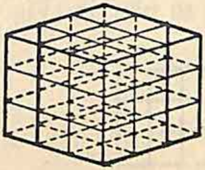


FIG. 15

represent the three layers of the cube from front to back (Fig. 16). The object of the game is, not to be the first to obtain a line, but to obtain as many lines as possible. *AAA* is a line running from front to back of the cube along the left-hand top edge. *BBB* is a line running diagonally

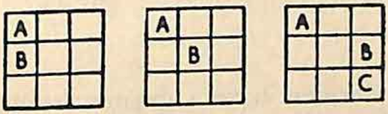


FIG. 16

from left to right across the central horizontal layer of the cube. *ABC* is a diagonal of the cube.

In the early stages of this game, the cube has to be visualized, and also the positions and directions of lines in it. After a time, however, a pattern of behaviour is seen and visualization is no longer needed. (For a straight line, any movement from the first to second layer must be repeated from the second to third layers. Note $B \rightarrow B \rightarrow B$, $A \rightarrow B \rightarrow C$.)

If a pupil acquires this degree of facility in the game, he might be introduced to four-dimensional noughts and crosses. In Fig. 16, a horizontal arrangement is used for the three layers. A vertical spacing serves equally well (Fig. 17). Games are played using this spacing for a little while. The two spacings are then combined to give nine squares to play in (Fig. 18). The game is played exactly

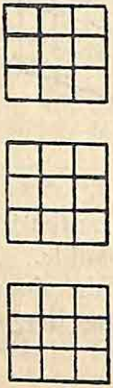


FIG. 17

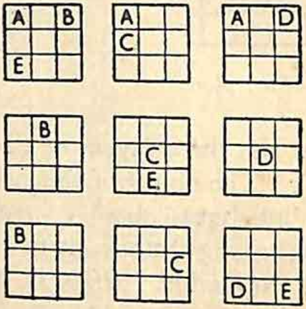


FIG. 18

as before, but we now have a greater freedom of movement. The shape which the squares represent cannot be visualized since it possesses four dimensions. The straight lines *AAA*, *BBB*, *CCC*, *DDD*, *EEE*, are shown, *ACE* is another straight line.

This is a true example of the mathematician's fourth dimension. A step which carries us from two dimensions to three dimensions is repeated to take us from three dimensions to four dimensions. It can be repeated if we wish. Noughts and crosses in five dimensions can be played on three sets of nine squares.

# BIBLIOGRAPHY

## TEACHING OF MATHEMATICS

*A Study of Mathematical Education*, Branford. Oxford University Press

*The Teaching of Algebra*, Nunn. Longman

*Diagnosis and Remedial Teaching in Arithmetic*, Schonell. Oliver and Boyd

*The Teaching of Mathematics*, I.A.A.M. Cambridge University Press

Reports of the Mathematical Association (published by Bell):

- *The Teaching of Algebra*
- *The Teaching of Arithmetic*
- *The Teaching of Geometry* (Second Report)
- *The Teaching of Trigonometry*
- *The Teaching of Mathematics in Primary Schools*
- *Mathematics in Secondary Modern Schools*

*Multi-Sensory Aids in the Teaching of Mathematics*, 18th Yearbook of the National Council of Teachers of Mathematics. Columbia University, New York

Also the publications of A.T.A. (Association for Teaching Aids in Mathematics); details from Mrs Fyfe, 91 Randolph Avenue, London W. 9

*Guidebook on the Teaching of Mathematics*, Narasimhachari. Government of Madras

*Technique of Teaching Mathematics*, Breslich. Chicago University Press

*Problems in Teaching Secondary School Mathematics*, Breslich. Chicago University Press

*The Teaching of Secondary School Mathematics*, Butler & Wren. McGraw-Hill

*Teaching of Elementary Mathematics*, Godfrey & Siddons. Cambridge University Press

*Teaching of Arithmetic*, Klapper. Longman

*Teaching of Algebra*, Durrell. Macmillan

*Kanitham Karpithal* (Tamil) (Teaching of Mathematics), Parthasarathy. Saiva Siddanta Noorpathippu Kazhagam

*Teaching of Mathematics in New Education*, Kuppuswami Iyengar. Doab
*Teaching of Arithmetic*, Montieth. Oxford University Press
*Teaching of Elementary Mathematics*, Smith. Macmillan
*Introducing Mathematics*, Sawyer. Penguin

1. Vision in Elementary Mathematics
2. The Search for Pattern
3. A Path to Modern Mathematics
4. Prelude to Mathematics

## GENERAL MATHEMATICS

*Mathematics, its Magic and Mastery*, Bakst. Chapman & Hall
*Polish up your Mathematics*, Fawdry. Bell
*Introduction to Mathematics*, Hill and McHarrie. Oxford University Press
*Mathematics for the Million*, Hogben. Allen and Unwin
*Mathematics in Theory and Practice*, Sawyer. Odham
*Mathematical Snapshots*, Steinhaus. Oxford University Press
*Tutorial Arithmetic*, Workman. University Tutorial Press
*Introduction to Mathematics*, Whitehead (Home University Library) Oxford University Press
*School Mathematics*, Breslich. Chicago University Press
*Mathematics Made Simple*, Sperling & Stuart. W. H. Allen
*Learning Mathematics*, Heritage. Penguin
*Modern Arithmetic*, Narasimhachari. Oxford University Press

## APPLICATIONS, ETC., OF MATHEMATICS

### 1. Astronomy

*Navigation and Astronomy for Students*, Allison. Harrap
*Elementary Mathematical Astronomy*, Barlow and Bryan. University Tutorial Press
*Guide to the Sky*, Beet. Cambridge University Press
*Observing the Heavens*, Hood. Oxford University Press
*Golden Book of Astronomy*, Wyler and Ames. Publicity Products
*Stars at a Glance*. (A cheap star atlas.) Philip
*The Universe Around Us*, Sir James Jeans. Penguin

### 2. Geography, Navigation, Surveying, etc.

*Mapwork and Practical Geography*, Bygott. University Tutorial Press.

*Mapmaking*, Debenham. Blackie.
*About Maps*, Hood. Penguin
*How Time is Measured*, Hood. Oxford University Press
*Experimental Geography*, Vols. I-III, Merriman. Nelson
*Surveying for Schools*, Richardson. Philip
*Practical Geography*, Simmons and Richardson. Macmillan
*Nautical Mathematics*, Walling and Hill. Cambridge University Press
*History of Surveying Instruments*, 19th Yearbook of the National Council of Teachers of Mathematics. Columbia University, New York

3. Drawing, Geometry, etc.

*Mathematical Models*, Cundy and Rollett. Oxford University Press
*Technical Drawing for Secondary Schools*, Books I-III, Evans and Williams. Newnes
*Descriptive Geometry*, Lee and Reekie. Arnold
*Pictorial Drawing for Engineers*, Parkinson. Pitman

4. Miscellaneous

*Practical Statistics*, Makin. Gee
*Designing and Making*, Sawyer. Blackwell
*A Mathematician Explains*, Logsdon. Chicago University Press
*Mathematician's Delight*, Sawyer. Penguin

## HISTORY OF MATHEMATICS

*Romance in Arithmetic*, Bowman. University of London Press
*Pyramids of Egypt*, Edwards. Penguin
*Short History of Mathematics*, Sanford. Harrap
*History of Mathematics, Vol. II*, Smith. Ginn
*Number Stories of Long Ago*, Smith. Ginn
*Story of Measurement, Books I-IV*, Smith. Blackwell
*Stories of Mathematics*, Williams. Evans
*Story of Reckoning in the Middle Ages*, Yeldham. Harrap
*The Teaching of Arithmetic through 400 Years*, Yeldham. Harrap
*History of Mathematics*, Cajory. Macmillan

*Short Account of the History of Mathematics*, Ball. Constable
*History of Mathematics*, Fint. Macmillan
*The Great Mathematicians*, Turnbull. Methuen
*Men of Mathematics*, Bell. Penguin

DIVERSIONS

*Fun with Figures*, Clarke. Heinemann
*Amusements in Mathematics*, Dudeney. Nelson
*Number, Fun and Facts*, Friend. Scribner
*Mathematical Recreations*, Kraitchik. Allen and Unwin
*Odd Numbers*, McKay. Cambridge University Press
*Mathematical Recreations*, Rouse Ball. Macmillan
*Mathematical Diversions*, Hunter. Van Nostrand
*Playing with Infinity*. Peter. G. Bell
*Mathematical Puzzles and Diversions*, Martin Gardner. G. Bell
*More Mathematical Puzzles and Diversions*, Martin Gardner. G. Bell
*Mathematics in your World*, Karl Menninger. G. Bell
*Trachtenburg* (The speed system of Basic Mathematics) Cutler & McShane. Fawcett
*Fun with New Mathematics*, Meyer & Hanlov. Pan
*Riddles in Mathematics*, Eugene Northrop. Penguin

REFERENCE

*Southern Railways Time Table*
*India-A Reference Annual*, Publications Division, Government of India
*Hindustan Year Book*, Sircar
*Current Affairs*, Sircar
*The Manorama Year Book*
*The Times of India Year Book*

JOURNALS

*The Mathematics Teacher* (National Council of Teachers of Mathematics)
*The Arithmetic Teacher* (National Council of Teachers of Mathematics)
*Teaching* (N.C.E.R.T)
*Mathematics Teacher* (S.I.T.U. Research Council)
*IAPL-Newsletter* (Indian Association for Programmed Learning)

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